Reduced-Complexity Widely Linear Adaptive Estimation

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***Abstract*—Widely linear filters play an important role in signal processing applications where the circularity properties on the complex data do not hold. They are able to achieve smaller mean-square error (MSE) than linear complex filters, but at a significantly higher computational cost. In this paper, we propose a modified version of widely linear filters with a reduced computational complexity. In the proposed version, the data vector is real, being constituted by the real and imaginary parts of the complex data separately. We prove that the new scheme achieves the same minimum MSE of standard widely linear estimators. We exemplify this idea for the least-mean squares (LMS) algorithm and also for the recursive least-squares (RLS) algorithm.**

***Index Terms*—Complex-valued signal processing, widely linear, adaptive filtering, LMS algorithm, RLS algorithm.**

I. Introduction

Widely Linear (WL) algorithms are employed to take into account the full second-order information from a complex signal, when circularity assumptions do not hold [1]–[5]. In this context, different WL adaptive algorithms have been proposed, as the algorithms based on the least-mean squares (LMS) algorithm [5]–[7], on the recursive least squares (RLS) algorithm [8], [9], and on the affine projection algorithm [10]. These WL algorithms are able to achieve a smaller mean-square error (MSE) in the steady-state than that of the *strictly linear* (SL) algorithms. Therefore, they have been successfully used in several applications, as, e.g., complex data prediction for wind forecast [6], suppression of interference in direct sequence code division multiple access (DS-CDMA) systems [9] and adaptive beamforming systems [5], [8].

The WL approach has a drawback when compared to traditional SL algorithms: the complex WL regressor vector is twice as long as the traditional SL vector, since the WL regressor consists of a concatenation of the SL regressor and its conjugate. This results in an increase of the computational complexity, an increase in the *excess* mean-square error, and possibly also (for LMS algorithms) a reduction in convergence speed. All these problems may overshadow the possible estimation gains for some applications.

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In this paper, we propose a reduced complexity version of the widely linear LMS and RLS algorithms. For this purpose, we introduce a real regressor vector composed of the concatenation of the real and the imaginary parts of the complex data. With this simple modification, the computational complexity of WL filters is reduced back to almost the complexity of strictly linear filters. We prove that the modified filters are equivalent to standard WL filters, only with a reduced complexity, and give a few examples for the modified WL-LMS and WL-RLS algorithms.

The paper is organized as follows. In Section II we make a brief introduction to the WL approach. Section III presents the proposed algorithms and Section IV provides a few examples. Conclusions are presented in Section V.

II. Standard WL estimation

Given a desired sequence $d(n)$ and a data vector $\mathbf{s}(n)$, the mean square estimation problem is to obtain the best (in terms of minimum variance of the error) linear estimate of $d(n)$ given $\mathbf{s}(n)$. The *strictly linear* approach to solve this problem in the case of complex variables is to minimize the cost-function

$$J(\mathbf{w}_{\text{SL}}) = \text{E}\{|e(n)|^2\} = \text{E}\{|d(n) - y(n)|^2\},$$

where

$$y(n) = \mathbf{w}_{\text{SL}}^H \mathbf{s}(n) \tag{1}$$

is the estimate, $\mathbf{s}(n)$ is the regressor vector, $\text{E}\{\cdot\}$ represents the expectation operator and $(\cdot)^H$ stands for the conjugate transpose of a vector or a matrix. For a given, fixed weight vector $\mathbf{w}_{\text{SL}}$, $J(\mathbf{w}_{\text{SL}})$ is the corresponding mean-square error (MSE). The optimum $\mathbf{w}_{\text{SL}}$ that minimizes the MSE, as is well-known, is given by the *Wiener solution*

$$\mathbf{w}_{\text{SL,opt}} = \mathbf{C}_{\text{ss}}^{-1} \boldsymbol{\rho}_{\text{SL}},$$

which is a $\text{M} \times 1$ vector. The solution depends on the input autocorrelation matrix $\mathbf{C}_{\text{ss}} = \text{E}\{\mathbf{s}(n)\mathbf{s}^H(n)\}$ and the cross-correlation vector $\boldsymbol{\rho}_{\text{SL}} = \text{E}\{d(n)\overline{\mathbf{s}}(n)\}$. An adaptive algorithm that iteratively computes an estimate for $\mathbf{w}_{\text{SL,opt}}$ is the least-mean squares algorithm, whose recursion is given by

$$\mathbf{w}_{\text{SL}}(n+1) = \mathbf{w}_{\text{SL}}(n) + \mu_{\text{SL}} e^*(n)\mathbf{s}(n),$$

with $e(n) = d(n) - \mathbf{w}_{\text{SL}}^H(n)\mathbf{s}(n)$, and where μ_{SL} is a step-size (a real constant). Note that this recursion requires 4M real multiplications and 4M real sums to evaluate $e(n)$, two real multiplications (to evaluate $\mu_{\text{SL}}e^*(n)$), 4M real multiplications and 2M sums (to evaluate $\big(\mu_{\text{SL}}e^*(n)\big)\mathbf{s}(n)$), 2M sums to update the weight vector, a total of $8\text{M}+2$ real multiplications and 8M sums.

Picinbono and Chevalier [1] seem to have been the first to notice that one could do better. In fact, full second-order information about $\mathbf{s}(n) = \mathbf{s}_{\text{R}}(n) + j\mathbf{s}_{\text{I}}(n)$ requires knowledge of three matrices, namely $\mathbf{R}_{\text{RR}} = \text{E}\{\mathbf{s}_{\text{R}}(n)\mathbf{s}_{\text{R}}^T(n)\}$, $\mathbf{R}_{\text{II}} = \text{E}\{\mathbf{s}_{\text{I}}(n)\mathbf{s}_{\text{I}}^T(n)\}$ and the cross-term $\mathbf{R}_{\text{RI}} = \text{E}\{\mathbf{s}_{\text{R}}(n)\mathbf{s}_{\text{I}}^T(n)\}$ (note that $\mathbf{R}_{\text{IR}} = \mathbf{R}_{\text{RI}}^T$.) However, this information cannot be retrieved from $\mathbf{C}_{\text{ss}}$ alone, since

$$\mathbf{C}_{\text{ss}} = \text{E}\{\mathbf{s}(n)\mathbf{s}^H(n)\} = (\mathbf{R}_{\text{RR}} + \mathbf{R}_{\text{II}}) + j(\mathbf{R}_{\text{RI}} - \mathbf{R}_{\text{IR}}). \quad (2)$$

That is, from $\mathbf{C}_{\text{ss}}$ we have only the sum of $\mathbf{R}_{\text{RR}}$ and $\mathbf{R}_{\text{II}}$, and the sum of $\mathbf{R}_{\text{RI}}$ with its transpose, from which the individual matrices cannot in general be recovered. They proposed that instead of using the strictly linear approach outlined above, one use an augmented vector $\overline{\mathbf{s}}(n)$ composed by the SL regressor vector $\mathbf{s}(n)$ and its conjugate, i.e.,

$$\overline{\mathbf{s}}(n) = \begin{bmatrix} \mathbf{s}(n) \\ \mathbf{s}^*(n) \end{bmatrix}.$$

In this case, two matrices are used to describe the second-order statistics of $\mathbf{s}(n)$: the covariance matrix $\mathbf{C}_{\text{ss}}$, defined as in the SL case, and its complementary covariance matrix $\boldsymbol{\Gamma}_{\text{ss}} = \text{E}\{\mathbf{s}(n)\mathbf{s}^T(n)\}$ [3]. $\boldsymbol{\Gamma}_{\text{ss}}$ can also be described in terms of the relations between the real and imaginary parts of $\mathbf{s}(n)$, as

$$\boldsymbol{\Gamma}_{\text{ss}} = (\mathbf{R}_{\text{RR}} - \mathbf{R}_{\text{II}}) + j(\mathbf{R}_{\text{RI}} + \mathbf{R}_{\text{IR}}). \quad (3)$$

From both (2) and (3), we now have full second-order information about $\mathbf{s}(n)$, since we can recover $\mathbf{R}_{\text{RR}}$, $\mathbf{R}_{\text{II}}$ and $\mathbf{R}_{\text{RI}}$ from $\mathbf{C}_{\text{ss}}$ and $\boldsymbol{\Gamma}_{\text{ss}}$. Algorithms based on $\overline{\mathbf{s}}(n)$ instead of $\mathbf{s}(n)$ are called *widely linear*.

Using $\mathbf{C}_{\text{ss}}$ and $\boldsymbol{\Gamma}_{\text{ss}}$, the covariance matrix for WL estimation can be written as

$$\mathbf{C}_{\overline{\text{ss}}} = \text{E}\{\overline{\mathbf{s}}(n)\overline{\mathbf{s}}^H(n)\} = \begin{bmatrix} \mathbf{C}_{\text{ss}} & \boldsymbol{\Gamma}_{\text{ss}} \\ \boldsymbol{\Gamma}_{\text{ss}}^H & \mathbf{C}_{\text{ss}}^H \end{bmatrix},$$

and using an augmented cross-correlation expectation vector as defined in [7]

$$\boldsymbol{\rho}_{\text{WL}} = \text{E}\{d(n)\overline{\mathbf{s}}(n)\} = \begin{bmatrix} \text{E}\{d(n)\mathbf{s}(n)\} \\ \text{E}\{d(n)\mathbf{s}^*(n)\} \end{bmatrix} = \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix},$$

the optimum WL solution is given by

$$\mathbf{w}_{\text{WL,opt}} = \mathbf{C}_{\overline{\text{ss}}}^{-1}\boldsymbol{\rho}_{\text{WL}},$$

where $\mathbf{C}_{\overline{\text{ss}}}$ is a $2\text{M} \times 2\text{M}$ matrix, while $\boldsymbol{\rho}_{\text{WL}}$ is a 2M column vector. Intuitively, we expect that WL-based estimators will achieve better MSE than the SL estimators, since they use full information. In fact, [1] and [3] show that WL estimation achieves at least the same MSE as its SL counterpart. SL estimators are optimum only for jointly circular processes (that is, $\boldsymbol{\Gamma}_{\text{ss}} = \mathbf{0}$ and $\mathbf{q} = \mathbf{0}$).

WL adaptive algorithms are direct extensions of their SL counterparts, only with twice as large regressors and weight vectors. Tables I and II present the WL version of the LMS and the RLS algorithms. In Table II, $\mathbf{P}_{\text{WL}}(n)$ is the estimate of the inverse covariance matrix, while $\mathbf{k}_{\text{WL}}(n)$ is the Kalman gain and $\gamma(n)$ is the conversion factor. λ is the forgetting factor and δ is a small constant for initializing $\mathbf{P}_{\text{WL}}(n)$. Note that the number of computations for WL filters is twice as large for LMS filters, and approximately four times larger for RLS filters (see Tables III and IV.)

TABLE I
WL-LMS ALGORITHM

Initialization
$\mathbf{w}_{\text{WL}}(0) = \mathbf{0}_{2\text{M}\times 1}$

for $\text{n} = 0, 1, 2, 3...$
$y(n) = \mathbf{w}_{\text{WL}}^H(n)\overline{\mathbf{s}}(n)$
$e(n) = d(n) - y(n)$
$\mathbf{w}_{\text{WL}}(n+1) = \mathbf{w}_{\text{WL}}(n) + \mu_{\text{WL}}e^*(n)\overline{\mathbf{s}}(n)$
end

TABLE II
WL-RLS ALGORITHM

Initialization
$\mathbf{w}_{\text{WL}}(0) = \mathbf{0}_{2\text{M}\times 1}$
$\mathbf{P}_{\text{WL}}(0) = \delta\mathbf{I}_{2\text{M}\times 2\text{M}}$

for $\text{n} = 0, 1, 2, 3...$
$\gamma(n) = (\lambda + \overline{\mathbf{s}}^H(n)\mathbf{P}_{\text{WL}}(n)\overline{\mathbf{s}}(n))^{-1}$
$\mathbf{k}_{\text{WL}}(n) = \mathbf{P}_{\text{WL}}(n)\overline{\mathbf{s}}(n)\gamma(n)$
$e(n) = d(n) - \mathbf{w}_{\text{WL}}^H(n)\overline{\mathbf{s}}(n)$
$\mathbf{w}_{\text{WL}}(n+1) = \mathbf{w}_{\text{WL}}(n) + \mathbf{k}_{\text{WL}}(n)e^*(n)$
$\mathbf{P}_{\text{WL}}(n+1) = \frac{1}{\lambda}(\mathbf{P}_{\text{WL}}(n) - \mathbf{k}_{\text{WL}}(n)\mathbf{k}_{\text{WL}}^H(n)\gamma(n))$
end

III. REDUCED-COMPLEXITY WL ESTIMATION

The same information contained in the 2M-complex regressor vector can be obtained if we use a 2M-real regressor, reducing the WL complexity, as we explain below.

Considering (2) and (3), we can observe that the information contained in the matrices $\mathbf{R}_{\text{RR}}$, $\mathbf{R}_{\text{II}}$ and $\mathbf{R}_{\text{RI}}$ is repeated in $\mathbf{C}_{\text{ss}}$ and in $\boldsymbol{\Gamma}_{ss}$ and, consequently, in $\mathbf{C}_{\overline{\text{ss}}}$. This argument intuitively leads to the idea of obtaining an equivalent real matrix for $\mathbf{C}_{\overline{\text{ss}}}$, avoiding redundance. It is possible to redefine the regressor vector in terms of its real and imaginary parts, i. e.,

$$\overline{\mathbf{s}}_{\text{RC}}(n) = \begin{bmatrix} \mathbf{s}_{\text{R}}(n) \\ \mathbf{s}_{\text{I}}(n) \end{bmatrix}. \quad (4)$$

In this case, the RC covariance matrix is given by

$$\mathbf{C}_{\text{RC}} = \begin{bmatrix} \mathbf{R}_{\text{RR}} & \mathbf{R}_{\text{RI}} \\ \mathbf{R}_{\text{IR}} & \mathbf{R}_{\text{II}} \end{bmatrix}.$$

Since $\bar{\mathbf{s}}_{\mathrm{RC}}(n)$ and $\mathbf{C}_{\mathrm{RC}}$ contain the same information as $\bar{\mathbf{s}}(n)$, one would expect that algorithms based on $\bar{\mathbf{s}}_{\mathrm{RC}}(n)$ should achieve the same MSE as WL filters. However, the computational cost will be significantly lower: since the regressor is now real, an LMS algorithm based on $\bar{\mathbf{s}}_{\mathrm{RC}}(n)$ will replace several multiplications of two complex numbers by multiplications of a complex and a real, which require each 2 multiplications and 2 sums less. As we show in Sections III-A and III-B, the total number of operations for the modified WL-LMS algorithm is the same as for the SL-LMS algorithm. For the modified WL-RLS algorithm, the number of operations is only slightly larger than that of the SL-RLS (see tables III and IV.)

We now prove that the algorithms obtained with the modified regressor achieves the same performance as the standard WL algorithms. The modified regressor $\bar{\mathbf{s}}_{\mathrm{RC}}(n)$ can be obtained from $\bar{\mathbf{s}}(n)$ applying a transformation

$$\mathbf{U} = \begin{bmatrix} \mathbf{I}_{\mathrm{M}\times\mathrm{M}} & j\mathbf{I}_{\mathrm{M}\times\mathrm{M}} \\ \mathbf{I}_{\mathrm{M}\times\mathrm{M}} & -j\mathbf{I}_{\mathrm{M}\times\mathrm{M}} \end{bmatrix}$$

to the vector $\bar{\mathbf{s}}(n)$, i.e.,

$$\mathbf{s}_{\mathrm{RC}}(n) = \mathbf{U}^H \bar{\mathbf{s}}(n) \tag{5}$$

so that

$$\mathbf{C}_{\mathrm{RC}} = \mathbf{U}^H \mathbf{C}_{\bar{\mathbf{s}}\bar{\mathbf{s}}} \mathbf{U}, \tag{6}$$

where $\mathbf{I}_{\mathrm{M}\times\mathrm{M}}$ is the identity. It should be noticed that $\mathbf{U}\mathbf{U}^H = 2\mathbf{I}$. [3] defined $\mathbf{U}$ (but normalized by a factor $1/\sqrt{2}$ to become unitary), but used it to analyze the standard WL estimation, not to reduce its computational complexity, as we do here.

Now, applying $\mathbf{U}$ to (1), we obtain

$$y(n) = \frac{1}{2}\mathbf{w}_{\mathrm{WL}}^H \mathbf{U}\mathbf{U}^H \bar{\mathbf{s}}(n).$$

Defining

$$\mathbf{w}_{\mathrm{RC}} = \frac{1}{2}\mathbf{U}^{\mathbf{H}}\mathbf{w}_{\mathrm{WL}}, \tag{7}$$

we note that the estimates computed using the transformed regressor and weight vector are equivalent to those obtained with the original vectors, since

$$y(n) = \mathbf{w}_{\mathrm{RC}}^H \bar{\mathbf{s}}_{\mathrm{RC}}(n).$$

Similarly, the RC-WL Wiener solution is given by

$$\mathbf{w}_{\mathrm{RC,opt}} = \mathbf{C}_{\mathrm{RC}}^{-1}\boldsymbol{\rho}_{\mathrm{RC}} = \left(\mathbf{U}^H \mathbf{C}_{\bar{\mathbf{s}}\bar{\mathbf{s}}}\mathbf{U}\right)^{-1}\mathbf{U}^H\boldsymbol{\rho}_{\mathrm{WL}} = \frac{1}{2}\mathbf{U}^H\mathbf{w}_{\mathrm{WL,opt}},$$

where $\boldsymbol{\rho}_{\mathrm{RC}} = \mathrm{E}\{d(n)\bar{\mathbf{s}}_{\mathrm{RC}}(n)\}$ is the RC cross-correlation vector.

From this it is easy to see that the optimum MSE achieved using the modified vector is the same as that achieved by the optimum WL solution. Next, we will show transformed versions of the widely linear LMS and RLS algorithms, obtained with the matrix $\mathbf{U}$ in order to use real regressor vectors.

A. Reduced-complexity widely linear LMS (RC-WL-LMS)

Applying the transformation $\mathbf{U}$ to the WL-LMS equations (Table I) and using the real regressor vector $\bar{\mathbf{s}}_{\mathrm{RC}}(n)$, we obtain

$$y(n) = \mathbf{w}_{\mathrm{RC}}^H(n)\bar{\mathbf{s}}_{\mathrm{RC}}(n),$$

$$e(n) = d(n) - y(n)$$

and

$$\mathbf{w}_{\mathrm{RC}}(n+1) = \mathbf{w}_{\mathrm{RC}}(n) + \mu_{\mathrm{RC}} e^*(n)\bar{\mathbf{s}}_{\mathrm{RC}}(n),$$

with the initial condition

$$\mathbf{w}_{\mathrm{RC}}(0) = \mathbf{0}_{2\mathrm{M}\times 1}.$$

Note that the relation between the step-sizes $\mu_{\mathrm{RC}} = 2\mu_{\mathrm{WL}}$ comes from the transformation $\mathbf{U}$.

Indeed, $y(n)$ and $e(n)$ are exactly the same as in the standard WL version, since the transformation $\mathbf{U}$ does not affect them. Furthermore, the analysis applied to the WL-LMS algorithm in [7] can be extended to the reduced complexity version, giving information about the convergence, the step-size and the MSE, for example.

B. Reduced-complexity widely linear RLS (RC-WL-RLS)

Using (6), the inverse of the covariance matrix $\mathbf{C}_{\mathrm{RC}}$ can be written as

$$\mathbf{C}_{\mathrm{RC}}^{-1} = \mathbf{U}^{-1}\mathbf{C}_{\bar{\mathbf{s}}\bar{\mathbf{s}}}^{-1}\left(\mathbf{U}^H\right)^{-1}. \tag{8}$$

Using properties of the similarity transformation, i.e., $\mathbf{U}^{-1} = \mathbf{U}^H/2$ and $\left(\mathbf{U}^H\right)^{-1} = \mathbf{U}/2$, (8) can be rewritten as

$$\mathbf{C}_{\mathrm{RC}}^{-1} = \frac{\mathbf{U}^H}{2}\mathbf{C}_{\bar{\mathbf{s}}\bar{\mathbf{s}}}^{-1}\frac{\mathbf{U}}{2}. \tag{9}$$

Denoting $\mathbf{P}_{\mathrm{RC}}(n)$ the estimate of $\mathbf{C}_{\mathrm{RC}}^{-1}$, the following relation holds

$$\mathbf{P}_{\mathrm{RC}}(n) = \frac{\mathbf{U}^H}{2}\mathbf{P}_{\mathrm{WL}}(n)\frac{\mathbf{U}}{2}. \tag{10}$$

Therefore, since matrix $\mathbf{P}_{\mathrm{WL}}(n)$ is initialized as $\mathbf{P}_{\mathrm{WL}}(0) = \delta\mathbf{I}$ in the WL-RLS algorithm, to ensure the equivalence to the reduced-complexity version proposed here, the matrix $\mathbf{P}_{\mathrm{RC}}(n)$ must be initialized as $\mathbf{P}_{\mathrm{RC}}(0) = (\delta/2)\mathbf{I}$.

Analogously, applying the transformation $\mathbf{U}$ to the conversion factor and the Kalman gain, we arrive at

$$\gamma(n) = \left[\lambda + \bar{\mathbf{s}}^H(n)\frac{\mathbf{U}\mathbf{U}^H}{2}\mathbf{P}_{\mathrm{WL}}(n)\frac{\mathbf{U}\mathbf{U}^H}{2}\bar{\mathbf{s}}(n)\right]^{-1} \tag{11}$$

and

$$\mathbf{k}_{\mathrm{RC}}(n) = \frac{\mathbf{U}^H}{2}\mathbf{k}_{\mathrm{WL}}(n). \tag{12}$$

Now, using $\bar{\mathbf{s}}_{\mathrm{RC}}(n)$ and $\mathbf{w}_{\mathrm{RC}}(n)$ as defined in (4) and (7), we obtain the RC-WL-RLC, i.e.,

$$\begin{aligned}
\gamma(n) &= \left(\lambda + \bar{\mathbf{s}}_{\mathrm{RC}}^H(n)\mathbf{P}_{\mathrm{RC}}(n)\bar{\mathbf{s}}_{\mathrm{RC}}(n)\right)^{-1}, \\
\mathbf{k}_{\mathrm{RC}}(n) &= \mathbf{P}_{\mathrm{RC}}(n)\bar{\mathbf{s}}_{\mathrm{RC}}(n)\gamma(n), \\
e(n) &= d(n) - \mathbf{w}_{\mathrm{RC}}^H(n)\bar{\mathbf{s}}_{\mathrm{RC}}(n), \\
\mathbf{w}_{\mathrm{RC}}(n+1) &= \mathbf{w}_{\mathrm{RC}}(n) + \mathbf{k}_{\mathrm{RC}}(n)e^*(n),
\end{aligned}$$

and

$$\mathbf{P}_{\mathrm{RC}}(n+1) = \frac{1}{\lambda}(\mathbf{P}_{\mathrm{RC}}(n) - \mathbf{k}_{\mathrm{RC}}(n)\mathbf{k}_{\mathrm{RC}}^{H}(n)\gamma(n)).$$

Again, the transformation reduces the presence of complex numbers in the calculations, which results in an algorithm with smaller cost.

Tables III and IV show the computational cost of the SL, standard WL and RC-WL algorithms based on LMS and RLS approaches, in terms of the number of real sums ($+$), multiplications ($\times$) and divisions ($\div$). We assume that the SL regressor vector has length M.

TABLE III
COMPUTATIONAL COMPLEXITY FOR SL-LMS, WL-LMS AND RC-WL-LMS IN TERMS OF REAL OPERATIONS PER ITERATION

Algorithm	$+$	$\times$	$\div$
SL-LMS	$8\mathrm{M}$	$8\mathrm{M}+2$	-
WL-LMS	$16\mathrm{M}$	$16\mathrm{M}+2$	-
RC-WL-LMS	$8\mathrm{M}$	$8\mathrm{M}+2$	-

TABLE IV
COMPUTATIONAL COMPLEXITY FOR SL-RLS, WL-RLS AND RC-WL-RLS IN TERMS OF REAL OPERATIONS PER ITERATION

Algorithm	$+$	$\times$	$\div$
SL-RLS	$6\mathrm{M}^2+14\mathrm{M}-1$	$7\mathrm{M}^2+21\mathrm{M}+1$	1
WL-RLS	$24\mathrm{M}^2+28\mathrm{M}-1$	$28\mathrm{M}^2+42\mathrm{M}+1$	1
RC-WL-RLS	$6\mathrm{M}^2+11\mathrm{M}$	$8\mathrm{M}^2+14\mathrm{M}+1$	1

From Table III, we notice that the RC-WL-LMS and SL-LMS algorithms have the same complexity, which is almost two times smaller than that of the WL-LMS algorithm. Figure 1 shows the number of real sums and multiplications of the algorithms.

Similarly, Table IV presents the complexity results for RLS. For this comparison, we have taken advantage of the symmetry of $\mathbf{P}_{\mathrm{RC}}(n)$ to calculate only the elements in the main diagonal and above it. We have used the approach of [11], p.201. Surprisingly, the number of sums needed by the RC-WL-RLS algorithm is smaller than that of the other two algorithms. The number of real multiplications that is only a little higher than that of SL-RLS. Figure 2 compares these algorithms for different values of the regressor length.

IV. EXAMPLES

In order to compare the SL, WL and RC-WL algorithms, we show some simulations considering the identification system model used in [7]. We defined a random process $\mathbf{s}(n) = \sqrt{1-\alpha^2}\mathbf{s}_{\mathrm{R}}(n) + j\alpha\mathbf{s}_{\mathrm{I}}(n)$, where $\mathbf{s}_{\mathrm{R}}(n)$ e $\mathbf{s}_{\mathrm{I}}(n)$ are two uncorrelated real-valued Gaussian processes with zero mean. The factor α is chosen between 0 and 1 and when it is chosen as $1/\sqrt{2}$, the process becomes circular. The system coefficients were defined as $w_{\mathrm{opt,k}} = \beta(1 + cos(2\pi(k-3)/5) - j(1 + cos(2\pi(k-3)/10)))$, with $k = 1, 2, ..., 5$ and $\beta = 0.432$. The desired signal included a Gaussian noise with 20dB signal to noise ratio (SNR). The WL-LMS algorithm used a step-size $\mu_{\mathrm{WL}} = 0.04$. The RLS forgetting factor was chosen as

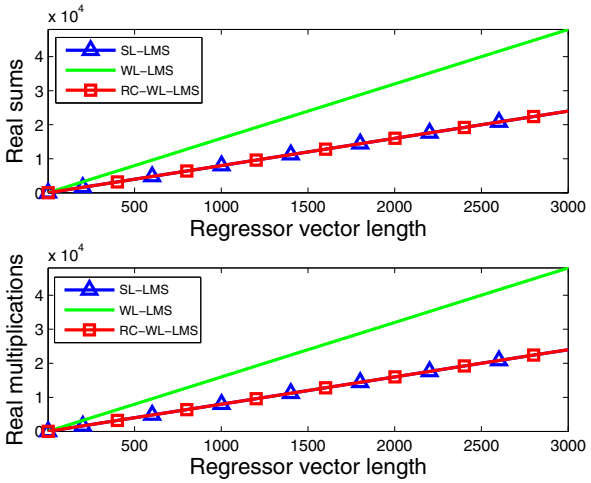

Fig. 1. Evolution of the number of real sums (top) and multiplications (down) with the increase of the regressor length for SL-LMS, WL-LMS and RC-WL-LMS

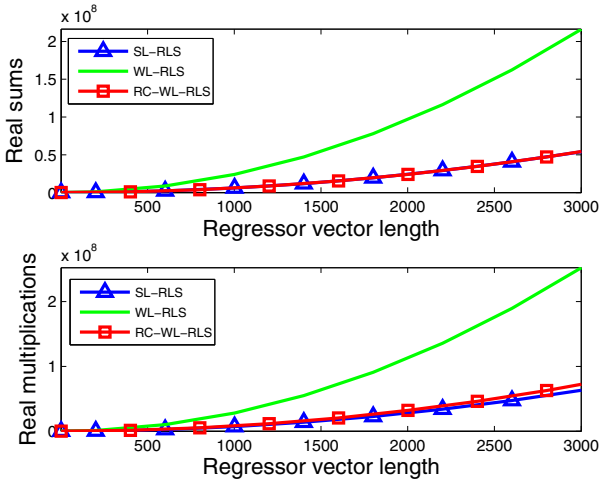

Fig. 2. Evolution of the number of real sums (top) and multiplications (down) with the increase of the regressor length for SL-RLS, WL-RLS and RC-WL-RLS

$\lambda = 0.999$ and the initial condition $\mathbf{P}_{\mathrm{WL}}(0) = 0.01\mathbf{I}$. The regressor was generated using both $\alpha = 0.1$ and $\alpha = 1/\sqrt{2}$. Note that we define $d(n) = \mathcal{R}\mathrm{e}\{\mathbf{w}_{\mathrm{opt}}^{H}\mathbf{s}(n)\}$, so the widely linear solution achieves an MSE that is better than that of the SL solution, even when the regressor is circular (since the regressor and desired sequences will not be jointly circular).

To ensure the same convergence characteristics, the step-size of the RC-WL-LMS was chosen as $\mu_{rc} = 2\mu_{wl} = 0.08$ and the RC-WL-RLS initial condition used for $\mathbf{P}_{\mathrm{RC}}(0)$ was $0.005\mathbf{I}$. In figures 3 and 4, we observe different versions of the LMS and RLS algorithms, considering the SL, WL and RC-WL estimation. These figures show that the WL and RC-WL estimators are clearly better than the SL approach when in this situation. It is also important to notice that the WL and RC-WL algorithms also achieve exactly the same MSE results,

reaffirming the equivalence between those approaches.

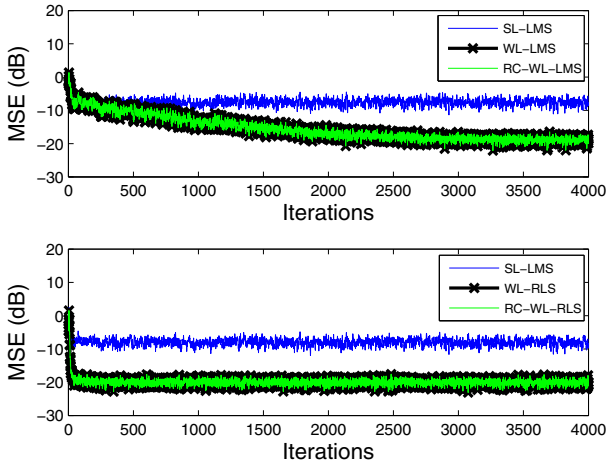

Fig. 3. MSE for WL and RC-WL algorithms with real noncircular input ($\alpha = 1/\sqrt{2}$). On the top: SL-LMS, WL-LMS and RC-WL-LMS ($\mu_{\mathrm{RC}} = 2\mu_{\mathrm{WL}} = 0.08$). On the bottom: SL-RLS, WL-RLS and RC-WL-RLS ($\mathbf{P}_{\mathrm{RC}}(0) = \mathbf{P}_{\mathrm{WL}}(0)/2 = 0.005\mathbf{I}$)

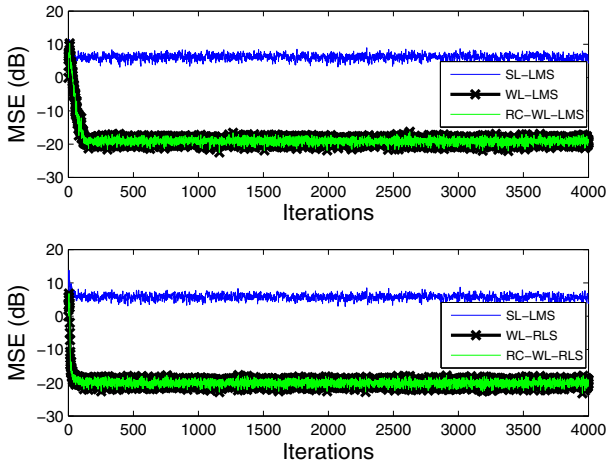

Fig. 4. MSE for WL and RC-WL algorithms with real circular input ($\alpha = 0.1$). On the top: SL-LMS, WL-LMS and RC-WL-LMS ($\mu_{\mathrm{RC}} = 2\mu_{\mathrm{WL}} = 0.08$). On the bottom: SL-RLS, WL-RLS and RC-WL-RLS ($\mathbf{P}_{\mathrm{RC}}(0) = \mathbf{P}_{\mathrm{WL}}(0)/2 = 0.005\mathbf{I}$)

V. Conclusion

We showed in this paper that using a real regressor vector constituted by the real and the imaginary parts of the complex data it is possible to reduce the number of operations in WL algorithms. The RC-WL-LMS achieved the same complexity as the SL-LMS algorithm, which is almost a 50% reduction compared to the standand WL-LMS. The RC-WL-RLS did not achieve the same number of operations as the SL-RLS, but the complexity reduction was also substantial. The simulations demonstrated the equivalence between the reduced-complexity algorithms and the WL ones when the input is circular or noncircular.

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