TRACKING ANALYSIS OF THE CONSTANT MODULUS ALGORITHM

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ABSTRACT

Recently, we proposed a model for the steady-state estimation error of real-valued constant-modulus-based algorithms as a function of the *a priori* error and of a term that measures the variability in the modulus of the transmitted signal. In this paper, we extend this model to complex-valued data and use it in conjunction with the feedback analysis method to obtain an analytical expression for the steady-state excess mean-square error (EMSE) of the Constant Modulus Algorithm (CMA). Such expression is more accurate for larger step-sizes than the previous ones in the literature, as confirmed by the good agreement between analytical and simulation results. Furthermore, from the EMSE expression, we obtain an estimate for the CMA step-size interval to ensure its convergence and stability, when it is initialized sufficiently close to the zero-forcing solution.

Index Terms— Adaptive filters, blind equalization, energy conservation, tracking analysis, Constant Modulus Algorithm.

1. INTRODUCTION

Blind equalizers are used in modern digital communication systems to remove intersymbol interference introduced by dispersive channels. The Constant Modulus Algorithm (CMA) [1] is the most popular for the adaptation of finite impulse response (FIR) equalizers due to its low computational complexity. Based on the link between blind and supervised equalization of [2], CMA can be interpreted as the blind version of the Least-Mean-Square (LMS) algorithm.

Analytical expressions for the steady-state excess mean-square error (EMSE) of constant-modulus-based algorithms have been computed in the literature (see, e.g., [3]-[6]). Using Lyapunov stability and averaging analysis, an approximate expression for the EMSE of CMA was obtained in [3]. Later, [4] and [5] focused on the CMA steady-state performance, using feedback analysis. Considering still the feedback method, [6] analyzed the tracking of constant-modulus-based algorithms in a unified manner. However, all these results are based on the assumption of a small step-size.

Recently, we proposed in [7] a model for the steady-state estimation error of constant-modulus-based algorithms as a function of the *a priori* error and of a term that measures the variability in the modulus of the transmitted signal. Such model, proposed for real-valued data, is based on the assumption that the optimum filter achieves perfect equalization [4, 5]. It also allows us to analyze the tracking performance of blind and supervised adaptive filters in a unified manner, since the variability in the transmitted signal modulus plays a role similar to the measurement noise in the supervised case.

This work was supported in part by FAPESP under grant 04/15114-2, and CNPq under grant 303.361/2004-2.

This paper has three main contributions. First, we extend the model of [7] to complex-valued data. Second, we use this model and the feedback method of [8] to derive an expression for the EMSE of CMA which is more accurate for larger step-sizes. Third, using this expression, we obtain an estimate for the step-size interval of CMA in order to ensure its convergence and stability, when it is initialized sufficiently close to the zero-forcing solution. To the best of our knowledge, simple bounds for the CMA step-size are not available in the literature. The paper is organized as follows. In Section 2, the problem is formulated. In Section 3, the steady-state analysis is presented. Simulation results and the conclusions are shown in sections 4 and 5, respectively.

2. PROBLEM FORMULATION

A simplified block diagram of a baseband communication system, considering a T/L fractionally-spaced equalizer (FSE) is depicted in Figure 1. Under certain well-known conditions, this model ensures perfect equalization in a noise-free environment, e.g. [9, 5]. The transmitted signal $a(n)$ is assumed i.i.d. (independent and identically distributed) and non Gaussian. We assume an M-tap FIR equalizer with input regressor vector $\mathbf{u}(n)$ and output $y(n) = \mathbf{u}^T(n)\mathbf{w}(n-1)$, where $\mathbf{w}(n-1)$ is the equalizer weight vector and the superscript T indicates transposition. The equalizer must mitigate the channel effects and recover the signal $a(n)$ for some delay τ_d. In blind equalization, there is no training data and the algorithms update $\mathbf{w}(n-1)$ using only higher-order statistics of the transmitted signal [1, 8].

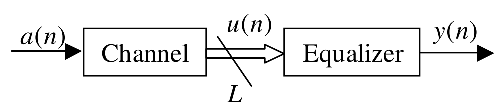

Fig. 1. Communication system model considering a T/L fractionally-spaced equalizer.

In this paper, we focus on the Constant Modulus Algorithm, whose update equation is given by

$$\mathbf{w}(n) = \mathbf{w}(n-1) + \mu e(n)\mathbf{u}^*(n), \tag{1}$$

where

$$e(n) = (r^a - |y(n)|^2)y(n), \tag{2}$$

μ is the step-size, $r^a \triangleq \mathrm{E}\{|a(n)|^4\}/\mathrm{E}\{|a(n)|^2\}$, the superscript $*$ stands for complex conjugate, and $\mathrm{E}\{\cdot\}$ is the expectation operator.

3. TRACKING ANALYSIS

We assume that in a nonstationary environment, the variation in the optimal solution $\mathbf{w}_o$ follows a random-walk model [8, p. 359], that is,

$$\mathbf{w}_o(n) = \mathbf{w}_o(n-1) + \mathbf{q}(n). \quad (3)$$

In this model, $\mathbf{q}(n)$ is an i.i.d. vector with positive-definite autocorrelation matrix $\mathbf{Q} = \mathrm{E}\{\mathbf{q}^*(n)\mathbf{q}^T(n)\}$ and is independent of the initial conditions $\{\mathbf{w}_o(-1), \mathbf{w}(-1)\}$ and of $\{\mathbf{u}(l)\}$ for all $l < n$ [8, Sec. 7.4]. For constant-modulus-based algorithms, there is more than one optimal solution since perfect equalization is achieved when the channel-equalizer impulse response assumes the form $[0 \cdots 0 \; e^{j\theta} \; 0 \cdots 0]^T$, where θ is a phase rotation. Since all these solutions give equally good results, we assume that the algorithm is initialized so that it converges to the case of $\theta = 0$. Since we are studying its steady-state performance, this does not imply a restriction in the applicability of our results.

One measure of the filter performance is given by the EMSE, defined as

$$\zeta \triangleq \lim_{n\to\infty} \mathrm{E}\{|e_a(n)|^2\}, \quad (4)$$

where $e_a(n) = \mathbf{u}^T(n)\widetilde{\mathbf{w}}(n-1)$ is the *a priori* error and $\widetilde{\mathbf{w}}(n-1) = \mathbf{w}_o(n-1) - \mathbf{w}(n-1)$ is the weight-error vector.

The steady-state analysis of CMA is based on the following assumptions:

A1. $\mathrm{E}\{a(n)|a(n)|^{k-1}\} = 0, \quad k = 2m+1, \; m \in \mathbb{N}$, and for complex data $\mathrm{E}\{a^2(n)\} = 0$ (circularity condition). In other words, $a(n)$ is sub-Gaussian and the constellation is symmetric, as is the case for most constellations used in digital communications [8].

A2. The signal-to-noise ratio at the input is high, so that $a(n-\tau_d) \approx \mathbf{u}^T(n)\mathbf{w}_o(n-1)$, i.e., the optimum filter achieves perfect equalization. However, due to channel variation and gradient noise, the equalizer weight vector $\mathbf{w}(n-1)$ is not equal to $\mathbf{w}_o(n-1)$, even in steady-state. Using the above approximation, we have $y(n) = \mathbf{u}^T(n)\mathbf{w}(n-1) = \mathbf{u}^T(n)[\mathbf{w}_o(n-1) - \widetilde{\mathbf{w}}(n-1)]$, i.e.,

$$y(n) \approx a(n-\tau_d) - e_a(n). \quad (5)$$

This approximation was also used in the CMA steady-state analyses of [4, Sec. III-A] and [5].

Using A2, the constant-modulus error of (2), can be rewritten as

$$e(n) = \gamma(n)e_a(n) + (3-\alpha)a^2(n-\tau_d)e_a^*(n) + \beta(n) + s(n), \quad (6)$$

where

$$\gamma(n) = \alpha|a(n-\tau_d)|^2 - r^a, \quad (7)$$

$$\beta(n) = a(n-\tau_d)r^a - a(n-\tau_d)|a(n-\tau_d)|^2, \quad (8)$$

$$s(n) = -2a(n-\tau_d)|e_a(n)|^2 - a^*(n-\tau_d)e_a^2(n) + e_a(n)|e_a(n)|^2, \quad (9)$$

and $\alpha = 2$ (resp., $\alpha = 3$) for complex (resp., real) data. If $e_a^2(n)$ is reasonably small in steady-state, terms depending on higher-order combinations of $e_a(n)$ can be disregarded in (6), i.e., $\lim_{n\to\infty} s(n) \approx 0$, which leads to the approximation

$$e(n) \approx \gamma(n)e_a(n) + (3-\alpha)a^2(n-\tau_d)e_a^*(n) + \beta(n). \quad (10)$$

From (8) and A1, $\beta(n)$ is an i.i.d. random variable, which satisfies $\mathrm{E}\{\beta(n)\} = 0$ and

$$\sigma_\beta^2 \triangleq \mathrm{E}\{|\beta(n)|^2\} = \mathrm{E}\{|a(n)|^6 - (r^a)^2|a(n)|^2\}. \quad (11)$$

Analogously, the first and second moments of $\gamma(n)$ are given respectively by

$$\bar{\gamma} \triangleq \mathrm{E}\{\gamma(n)\} = \alpha\mathrm{E}\{|a(n)|^2\} - r^a \quad (12)$$

and

$$\xi \triangleq \mathrm{E}\{\gamma^2(n)\} = \alpha(\alpha-2)\, r^a\, \mathrm{E}\{|a(n)|^2\} + (r^a)^2. \quad (13)$$

For sub-Gaussian constellations (see A1), we always have $\bar{\gamma} > 0$.

The model (10) was proposed in [7] for real-valued data and is extended here for complex-valued data. We should notice that the same model also holds for supervised filters, only in that case we would have $\gamma(n) \equiv 1$, $\alpha = 3$, and the measurement noise $v(n)$ instead of $\beta(n)$. In addition, $\beta(n)$ is identically zero for constellations which do have constant modulus, so the variability in the modulus of $a(n)$ (as measured by $\beta(n)$) plays the role of measurement noise for CMA.

To proceed, we also assume that

A3. $a(n-\tau_d)$ and $e_a(n)$ are independent in steady-state. This assumption essentially requires the steady-state output fluctuations $\{e_a(n)\}$ to be insensitive, in steady-state, to the actual transmitted symbols $\{a(n)\}$ [4, 5]. An immediate consequence of this assumption is that $\gamma(n)$ and $\beta(n)$ are also independent of $a(n-\tau_d)$ in steady-state.

A4. $\|\mathbf{u}(n)\|^2$ and $e_a(n)$ are independent in steady-state. This requires the energy of the input vector to be independent of the *a priori* error [4, *As.I.2*, p.84];

A5. $\mathrm{E}\{\|\widetilde{\mathbf{w}}(n)\|^2\} = \mathrm{E}\{\|\widetilde{\mathbf{w}}(n-1)\|^2\}$ when $n \to \infty$, i.e., the filters are operating in stable conditions, and have reached steady-state.

The update recursion (1) can be written in terms of the weight-error vector $\widetilde{\mathbf{w}}(n) = \mathbf{w}_o(n) - \mathbf{w}(n)$. Subtracting both sides of (1) from $\mathbf{w}_o(n)$ and using (3), we get

$$\widetilde{\mathbf{w}}(n) - \mathbf{q}(n) = \widetilde{\mathbf{w}}(n-1) - \mu e(n)\mathbf{u}^*(n). \quad (14)$$

By evaluating the energies, i.e, the squared Euclidean norms on both sides of (14), we obtain

$$\|\widetilde{\mathbf{w}}(n) - \mathbf{q}(n)\|^2 = \|\widetilde{\mathbf{w}}(n-1)\|^2 + \mu^2\|\mathbf{u}(n)\|^2|e(n)|^2 - \mu\left[e(n)e_a^*(n) + e^*(n)e_a(n)\right]. \quad (15)$$

By taking expectations of both sides of (15), using A4, A5, and the model (3) , we arrive at

$$-\mathrm{E}\{\|\mathbf{q}(n)\|^2\} \approx -\mu\mathrm{E}\{e(n)e_a^*(n) + e^*(n)e_a(n)\} + \mu^2\mathrm{E}\{\|\mathbf{u}(n)\|^2\}\mathrm{E}\{|e(n)|^2\}. \quad (16)$$

To simplify (16), we use (4) and the model (10) in conjunction with A1, A3, and A4 to get the following approximations:

$$\mathrm{E}\{|e(n)|^2\} \approx \left[\xi + (3-\alpha)\mathrm{E}\{|a(n)|^4\}\right]\zeta + \sigma_\beta^2 \quad (17)$$

and

$$\mathrm{E}\{e(n)e_a^*(n) + e^*(n)e_a(n)\} \approx 2\,\bar{\gamma}\,\zeta. \quad (18)$$

Replacing (17) and (18) in (16), we obtain

$$\zeta \approx \frac{\mu\,\sigma_\beta^2\,\mathrm{E}\{\|\mathbf{u}(n)\|^2\} + \mu^{-1}\mathrm{E}\{\|\mathbf{q}(n)\|^2\}}{2\bar{\gamma} - \mu\mathrm{E}\{\|\mathbf{u}(n)\|^2\}\left[\xi + (3-\alpha)\mathrm{E}\{|a(n)|^4\}\right]}. \quad (19)$$

The terms $\mathrm{E}\{\|\mathbf{u}(n)\|^2\}$ and $\mathrm{E}\{\|\mathbf{q}(n)\|^2\}$ can be replaced by the traces of matrices $\mathbf{R}$ and $\mathbf{Q}$, respectively, i.e., $\mathrm{Tr}(\mathbf{R})$ and $\mathrm{Tr}(\mathbf{Q})$, where $\mathbf{R} \triangleq \mathrm{E}\{\mathbf{u}^*(n)\mathbf{u}^T(n)\}$ is the autocorrelation matrix of the input regressor vector $\mathbf{u}(n)$. Then, we arrive at the analytical expression for the steady-state EMSE of CMA, shown in Table 1. The EMSE expression of the LMS algorithm can also be obtained from (19), making $\sigma_\beta^2 = \sigma_v^2$, $\alpha = 3$, $\bar{\gamma} = 1$, and $\xi = 1$. Thus, there is an evident equivalence between LMS and CMA, as we can observe comparing the expressions of Table 1.

Since the EMSE is positive by definition, the analytical expressions of Table 1 must always provide positive estimates. This can be used to obtain a rough estimate of the range of step-sizes for stable behavior of CMA. The denominator of (19) will be positive if the CMA step-size is chosen in the following interval

$$0 < \mu < \mu_{\max} = \frac{2\bar{\gamma}}{\mathrm{Tr}(\mathbf{R})\,[\xi + (3-\alpha)\mathrm{E}\{|a(n)|^4\}]}. \qquad (20)$$

For LMS, this argument leads to the range $0 < \mu < 2/\mathrm{Tr}(\mathbf{R})$, which is the same approximation provided in [10].

To close this section, we should notice that the model (10) also allows us to analyze the tracking performance of CMA using the traditional method, where one computes a recursion for the autocorrelation matrix of the weight-error vector. However, the feedback method is less laborious, allowing one to obtain good approximations in a more immediate manner [8, Ch. 7]. This justifies the choice of the feedback method in the analysis presented here.

Table 1. Analytical expressions for steady-state EMSE.

Alg.	ζ
CMA	$\dfrac{\mu\sigma_\beta^2\mathrm{Tr}(\mathbf{R}) + \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2\bar{\gamma} - \mu\mathrm{Tr}(\mathbf{R})\,[\xi + (3-\alpha)\mathrm{E}\{|a(n)|^4\}]}$
LMS	$\dfrac{\mu\sigma_v^2\mathrm{Tr}(\mathbf{R}) + \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2 - \mu\mathrm{Tr}(\mathbf{R})}$

4. SIMULATION RESULTS

To verify the validity of the tracking analysis for real and complex data, we assume $\mathbf{Q} = \sigma_q^2\mathbf{I}$, with $\sigma_q^2 = 10^{-6}$ and $\mathbf{I}$ the identity matrix. In the real data case, we use 4-PAM (Pulse Amplitude Modulation) with symbols $\{\pm 1, \pm 3\}$ and statistics $\mathrm{E}\{a^2(n)\} = 5$, $\mathrm{E}\{a^4(n)\} = 41$, and $\mathrm{E}\{a^6(n)\} = 365$. We consider the channel 1 of Table 2 and $M = 4$ coefficients for a $T/2$-FSE, which is initialized with only one non-null element in the second position. Figure 2 shows the EMSE estimated from the ensemble-average of 100 independent runs for each step-size μ. We reject the experiments in which the algorithm diverges or converges to local minima, since our analysis is valid for $e_a(n)$ sufficiently small (see assumption A2). We also show the theoretical values predicted by the expression of Table 1 and by the expression derived in [5]. For $\mu < 10^{-3}$, both expressions give almost the same values, which agree reasonably well with the simulation results. In this case, the step-size μ is small enough for the denominator of (19) to be well approximated by $2\bar{\gamma}$. However, for larger step-sizes ($\mu > 10^{-3}$, in this example), the approximation is no longer valid, and the expressions give different values. As can be seen from the figure, the values predicted by the expression of Table 1 are more accurate than those predicted by [5].

In the complex data case, we use 8-QAM (Quadrature Amplitude Modulation) with symbols $\{\pm 1, \pm j, +1 \pm j, -1 \pm j\}$ and statistics $\mathrm{E}\{|a(n)|^2\} = 1.5$, $\mathrm{E}\{|a(n)|^4\} = 2.5$, and $\mathrm{E}\{|a(n)|^6\} = 4.5$. We consider the channel 2 of of Table 2 and $M = 6$ coefficients for a $T/2$-FSE, which is initialized with only one non-null element in the third position. Figure 3 shows the EMSE estimated from the ensemble-average of 100 independent runs for each μ. Again, we reject the runs in which the algorithm diverges or converges to local minima. For $\mu < 10^{-2}$, our expression and the one derived in [5] give similar values, and good agreement with simulations. However, for larger step-sizes ($\mu > 10^{-2}$, in this example), the values predicted by the expression of Table 1 are again more accurate than those predicted by [5].

To verify the step-size interval of (20), we estimate the CMA's probability of divergence P_d [11], which is obtained from $N_{\exp}$ repetitions of each experiment, starting from the same initial condition $\mathbf{w}(0)$. As in [11], we assume that a sample function is labeled as "diverging", if $\|\mathbf{w}(N_{\mathrm{it}})\| \geq 10^4$ after N_{it} iterations. Then, we compute the probability of divergence as

$$P_d = (\text{Number of curves diverging})/N_{\exp}.$$

Figure 4 shows values of the estimated P_d as a function of $\mu = p_\mu\mu_{\max}$, $0 < p_\mu \leq 1$ for the real and complex-valued previous examples, but considering a stationary environment ($\sigma_q^2 = 0$) and an initialization close to the optimal solution. We can observe that the superior limit $\mu_{\max}$ is in the region where the algorithm becomes unstable with $P_d = 1$. Hence, to guarantee the stability of CMA, μ must be chosen smaller than $\mu_{\max}$, e.g., $\mu = 0.3\mu_{\max}$, where $P_d \approx 0$. This was also observed for a great variety of situations such as different channels and number of coefficients.

Although a step-size much smaller than $\mu_{\max}$ can ensure the stability of CMA, this result will be valid only if the initialization of the algorithm is sufficiently close to the optimal solution. Figure 5 shows the probability of divergence as a function of the initialization $\mathbf{w}(0) = p_w\mathbf{w}_\mathrm{o}$, $p_w > 0$, considering a constant step-size: $\mu = \mu_{\max}/3 = 0.0030$ for the real case and $\mu = 0.4\mu_{\max} = 0.0321$ for the complex case. Note that we initialize the algorithm with a vector in the same direction as the optimal solution, but with different magnitudes. The probability of divergence increases for $p_w > 1.5$, being more critical in the complex-valued example. For an initialization distant from the optimal solution but with a smaller magnitude (e.g., $p_w = 0.1$), although divergence was not observed, the algorithm did not always converge to the optimal solution, staying several times in local minima.

Table 2. Coefficients of the channels used in the simulations.

channel 1	channel 2
-0.1761	$-0.1761 - j0.1970$
-0.6166	$-0.6166 - j0.7580$
$+0.5943$	$+0.5943 + j0.0054$
-0.1080	$-0.1080 - j0.0193$
$+0.0505$	$+0.0505 + j0.0110$
$+0.0911$	$+0.0911 + j0.0246$

5. CONCLUSION

Using a steady-state model for the estimation error of constant-modulus-based algorithms in conjunction with the feedback method, we derived a more accurate analytical expression for the steady-state EMSE of CMA. It is equivalent to the EMSE expression of the LMS algorithm, which reinforces the link between blind and non-blind adaptive filters. Using this result, we estimate the steady-state EMSE of CMA equalizers and find it in reasonable agreement with experimental results. From the EMSE expression, we obtained a rough and easy to compute estimate of the range of step-sizes to guarantee the stability of CMA. A more accurate estimate will be presented in a future work.

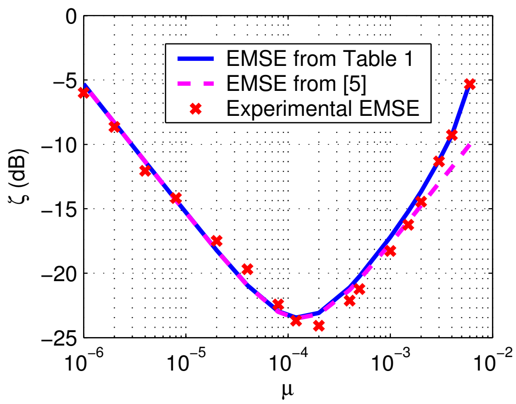

Fig. 2. Theoretical and experimental steady-state EMSE as a function of step-size for CMA; mean of 100 independent runs; 4-PAM, $M = 4$, $T/2$-FSE, channel 1, $\mathbf{Q} = 10^{-6}\mathbf{I}$.

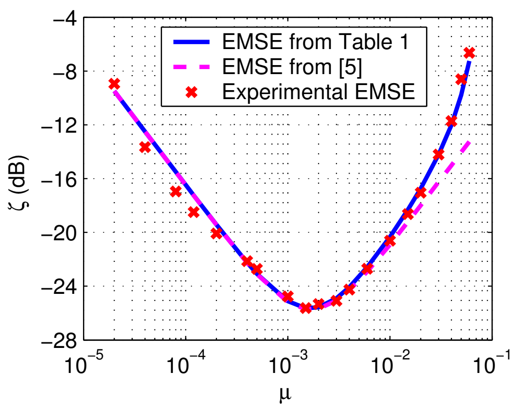

Fig. 3. Theoretical and experimental steady-state EMSE as a function of step-size for CMA; mean of 100 independent runs; 8-QAM, $M = 6$, $T/2$-FSE, channel 2, $\mathbf{Q} = 10^{-6}\mathbf{I}$.

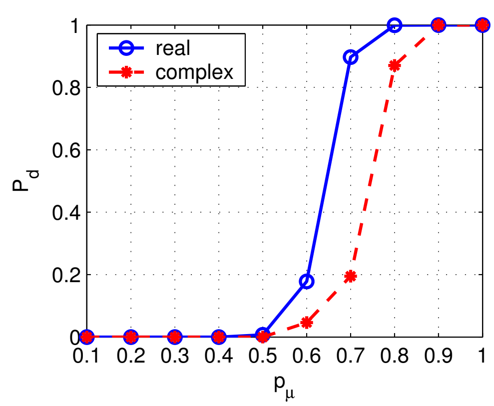

Fig. 4. Probability of divergence as a function of step-size $\mu = p_\mu \mu_{\max}$; $N_{\rm exp} = 10^3$, $N_{\rm it} = 10^4$. Real case: 4-PAM, $M = 4$, $T/2$-FSE, $\mu_{\max} = 0.0091$, $\mathbf{w}_{\rm o}^T = \mathbf{w}^T(0) = [-0.4748\ 1.5827\ -0.0497\ -0.7501]$. Complex case: 8-QAM, $M = 6$, $T/2$-FSE, $\mu_{\max} = 0.0801$, $\mathbf{w}_{\rm o}^T = \mathbf{w}^T(0) = [(0.0307 + j0.1548)\ (-0.5351 - j0.3482)\ (1.5493 + j0.0228)\ (-0.0005 - j0.0587)\ (0.2306 + j0.1247)\ (-0.8105 + j0.1354)]$.

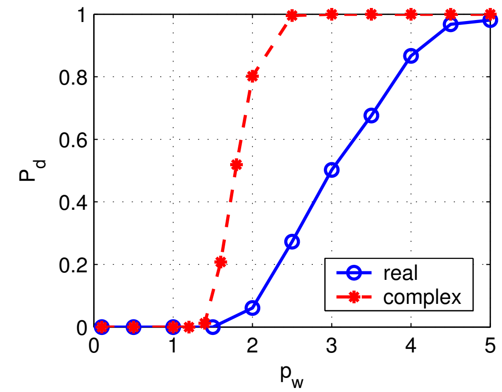

Fig. 5. Probability of divergence as a function of the initialization $\mathbf{w}(0) = p_w \mathbf{w}_{\rm o}$; $N_{\rm exp} = 10^3$, $N_{\rm it} = 10^4$. Real case: 4-PAM, $M = 4$, $T/2$-FSE, $\mu = \mu_{\max}/3 = 0.0030$. Complex case: 8-QAM, $M = 6$, $T/2$-FSE, $\mu = 0.4\mu_{\max} = 0.0321$; $\mathbf{w}_{\rm o}$ is the same as in Fig. 4.

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