# MODELING FINITE PRECISION LMS BEHAVIOR USING MARKOV CHAINS

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**ABSTRACT**

We propose a new model for the behavior of the Least Mean Square (LMS) algorithm when implemented in finite precision. We model the adaptive filter coefficients as a Markov chain and determine its transition probability matrix for the one-dimensional case. We also determine conditions to avoid the so-called stopping phenomenon. The proposed model eliminates the linearizations used in previous models, accounts for saturation effects and leads to accurate estimations of the mean-square error behavior. Monte Carlo simulation results illustrate the quality of the proposed model.

## 1. INTRODUCTION

The LMS (Least Mean Square) is one of the preferred algorithms for real-time adaptive system implementations because of its simplicity and robustness [1, 2]. Real-time implementations frequently use *hardware* that operates with fixed-point arithmetic. In these cases, infinite precision models become inadequate. The accumulation of quantization errors and other nonlinear effects inherent to finite precision operation can lead to behaviors that significantly deviate from theoretical predictions based on infinite precision models. Thus, it is of great interest to understand the behavior of the LMS algorithm when implemented in finite precision.

The finite precision behavior of the LMS algorithm has been studied by several authors [1]–[9]. In [3] and [4], analytical models were derived based on a linearized approximation of the quantization errors, which were modeled by additive white noise. The linearized model may adequate during the early stages of adaptation, when the error is large, and if saturation does not occur. After the initial acquisition period, the algorithm behavior can no longer be predicted by a linear model [8]. In [6]–[9], nonlinear models were derived which incorporate the nonlinear nature of the quantization in the weight update equation, leading to more accurate predictions of the algorithm behavior. Such models, however, still do not consider the saturation effects inherent to the finite precision processing.

In this paper we propose a new modeling technique for the behavior of the LMS algorithm. We model the adaptive weight vector as a state in a Markov chain, and study the signal statistics at a given iteration conditioned on the state at the previous iteration. We study the unidimensional case to provide insight on the new approach while keeping the mathematics simple. The new model accounts for all nonlinearities, including saturation effects.

The paper is divided in four parts. In Section 2 we define the problem and establish the notation. In Section 3 we determine the conditional probability density function (pdf) of the adaptive weight at iteration $n$, conditioned on the value of the weight at iteration $n-1$. In Section 4 conditions on the step size are given. In Section 5 we use the new model to study the mean-square error behavior. Finally, we provide simulation results that verify the accuracy of the theoretical model.

## 2. PROBLEM DEFINITION

Consider the system identification block diagram shown in Fig. 1.

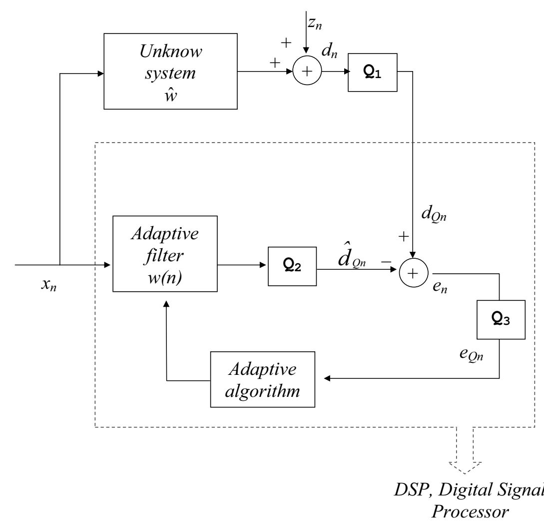


**Fig. 1**. System identification implemented in finite precision

In Fig. 1, $x_n$ is the input signal, $z_n$ is a zero-mean additive white Gaussian noise. $d_n$ is the desired signal. $\hat{w}$ is the unknown system response and $w_n$ is the adaptive filter coefficient. $\mathbf{Q_1}$, $\mathbf{Q_2}$ and $\mathbf{Q_3}$ are identical $b$-bit quantizers. We assume for simplicity that all finite precision signals and coefficients are quantized with $b$ bits. The uniform quantization step is then $\Delta = 2^{1-b}$ and the quantization limits are $[-1, 1-\Delta]$. Fig. 2 illustrates the transfer characteristic of the quantizers for $b = 3$. The input $x_n$ is a $b$-bit quantized discrete uniform random signal such that $-1 \le x_n \le 1-\Delta$. This is a good model for signals in digital transmission systems. $d_{Q_n}$ is the quantized version of $d_n$, $\hat{d}_{Q_n}$ is the $b$-bit quantized adaptive filter output[1]. $e_{Q_n}$ is the estimation error represented in $b$ bits. There are $N = 2^b$ levels at the quantizer outputs.

[1]In general, the output will have at least $2b$ bits before $\mathbf{Q_2}$, because of the larger accumulator word-length in digital processors.

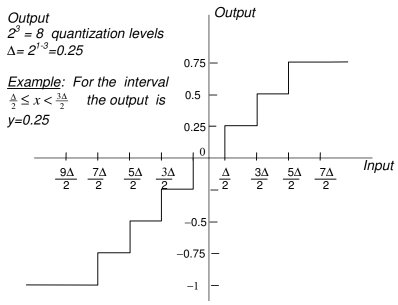


**Fig. 2**. Relation between input and output for a 3 bits quantizer

### 2.1. LMS algorithm in finite precision

The LMS weight update equation is given by

$$w_{n+1} = Q\{w_n + y_{Qn}\} \tag{1}$$

where

$$\begin{aligned} y_{Qn} &= Q\{\ \mu e_{Qn} x_n\ \} \\ e_{Qn} &= Q\{e_n\} \\ e_n &= d_{Qn} - \hat{d}_{Qn} \\ d_{Qn} &= Q\{\ \hat{w} x_n + z_n\ \} \\ \hat{d}_{Qn} &= Q\{\ w_n x_n\ \} \end{aligned} \tag{2}$$

The pdf of $x_n$ is given by

$$f_{\mathbf{x}}(x) = \frac{1}{N} \sum_{k=-2^{(b-1)}}^{k=2^{(b-1)}-1} \delta(x - k\Delta) \tag{3}$$

and the pdf of $z_n$ is $f_{\mathbf{z}}(z) = 1/\sqrt{2\pi}\sigma_z \ \exp(-z^2/2\sigma_z^2)$. Note that we need to define $e_{Q_n}$ and to include the quantization operation in the definition of $w_{n+1}$ because the values of $e_n$ and $w_n + y_{Qn}$ can exceed the saturation limits of the quantizer.

To study the dynamics of $w_n$, we next determine the pdf of $w_{n+1}$ conditioned in $w_n$.

## 3. STATISTICS OF THE ADAPTIVE WEIGHTS

Given (1) and (2), the statistics of $w_{n+1}$ depends on the statistics of $y_{Q_n}, e_{Q_n}, d_{Q_n}$ and $\hat{d}_{Q_n}$. We start by studying the statistics of $d_{Qn}$.

### 3.1. Statistics of $d_{Qn}$

From (1), $d_{Qn} = Q\{d_n\}$ with $d_n = \hat{w} x_n + z_n$. Given $\hat{w}$, we define $x_{o_n} = \hat{w} x_n$. Then, dropping the subscript $n$ for clarity, $\Pr\{\mathbf{x_o} = x_o\} = \Pr\{\hat{w} x = x_o\} = \Pr\{x = x_o/\hat{w}\}$ and

$$f_{\mathbf{x_o}}(x_o) \quad = \quad \frac{1}{N} \sum_{k=-2^{(b-1)}}^{k=(2^{b-1})-1} \delta(x_o - k\hat{w}\Delta) \tag{4}$$

Since $x_n$ and $z_n$ are independent random variables, the pdf of $d_n$ is given by convolution of the individual pdfs. Using (4) and $f_{\mathbf{z}}(z)$,

$$f_{\mathbf{d}}(d) \quad = \quad \frac{1}{N\sigma_z\sqrt{2\pi}} \sum_{k=-2^{(b-1)}}^{k=2^{(b-1)}-1} e^{-\left(\frac{d - \hat{w}k\Delta}{\sqrt{2}\sigma_z}\right)^2} \tag{5}$$

To determine the pdf of $d_{Q_n}$, we determine the probability that $d_n$ is in the $i$-th quantization interval $[d_{1_i}, d_{2_i}]$. Denoting this probability $D_i$, we have $D_i = \int_{d_{1_i}}^{d_{2_i}} f_{\mathbf{d}}(d)\mathrm{d}d$. Using (5) and integrating,

$$\begin{aligned} D_i = \frac{1}{N} \sum_{k=-2^{(b-1)}}^{k=2^{(b-1)}-1} \Bigg[ &\mathrm{erf}\left(\frac{d_{2_i} - k\hat{w}\Delta}{\sigma_z}\right) \\ &- \mathrm{erf}\left(\frac{d_{1_i} - k\hat{w}\Delta}{\sigma_z}\right)\Bigg] \end{aligned} \tag{6}$$

with $d_{1_i} = i\Delta - 0.5\Delta$ and $d_{2i} = i\Delta + 0.5\Delta$ (notice that the lower limit of the first interval is $-\infty$, and the upper limit of the last interval is $+\infty$). The pdf is then given by:

$$f_{\mathbf{d_Q}}(d_Q) = \sum_{i=-2^{(b-1)}}^{i=2^{(b-1)}-1} D_i \delta(d_Q - i\Delta) \tag{7}$$

We now proceed to determine the statistics of $e_{Qn}$.

### 3.2. Conditional pdf of $e_{Qn}$

From Fig. 1 and from (2), $e_{Q_n} = Q\{d_{Qn} - \hat{d}_{Qn}\}$, with $\hat{d}_{Qn} = Q\{w_n x_n\}$. Random variables $d_{Qn}$ and $Q\{w_n x_n\}$ depend on $x_n$ and on $w_n$. Then, the conditional pdf of interest is $f_{\mathbf{e_Q}}(e_Q|k\Delta, w_n)$.

Notice that $e_n$ in (2) can exceed the upper and lower quantization limits. Then, we must set $e_{Qn} = -1$ for $e_n < -1$ and $e_{Qn} = 1 - \Delta$ for $e_n > (1 - \Delta)$.

We first determine the pdf of $e_n$. Considering that $d_{Q\max} = \hat{d}_{Q\max} = 1 - \Delta$ and $d_{Q\min} = \hat{d}_{Q\min} = -1$, we have that $\{e_n\}\max = d_{Q\max} - \hat{d}_{Q\min} = 2 - \Delta$ and $\{e_n\}_{\min} = d_{Q\min} - \hat{d}_{Q\max} = -2 + \Delta$. The pdf of $e_n$ will then be

$$f_{\mathbf{e}}\left\{e|k\Delta, w_n\right\} = \sum_{k_e=-2^b+1}^{k_e=2^b-1} E_{k_e} \delta(e - k_e\Delta) \tag{8}$$

where $E_{k_e}$ is the probability of occurrence of the $k_e$-th state of $e_n$. To determine $E_{k_e}$, let $\hat{d}_{n_1} = \mathrm{round}\{w_n k\}\Delta$ where, for $m \in \mathbb{Z}$,

$$\mathrm{round}\{x\} = \lfloor x \rceil = \begin{cases} m, & x < m + \frac{1}{2} \\ m+1, & x \geq m + \frac{1}{2} \end{cases} \tag{9}$$

As $\hat{d}_{n_1}$ can exceed the quantizer limits when $w_n = -1$ and $x_n = -1$, we define $\hat{d}_n = \hat{d}_{n_1} - \Delta\delta(1 - \hat{d}_{n_1})$ and write $e_n = d_{Qn} - \hat{d}_n$. Then,

$$\begin{aligned} &\Pr\{e_n = e|k\Delta, w_n\} = \Pr\{d_{Qn} - \hat{d}_n|k\Delta, w_n\} = \\ &\Pr\{d_{Qn} = e + \hat{d}_n|k\Delta, w_n\} \end{aligned} \tag{10}$$

Using $\Pr\{d_{Qn}\}$ determined from (7), we have:

$$E_{k_e} = \frac{1}{N}\left[\mathrm{erf}\left(\frac{d_{2k_e} - \hat{w}x}{\sigma_z}\right) - \mathrm{erf}\left(\frac{d_{1k_e} - \hat{w}x}{\sigma_z}\right)\right]$$

with (again, excluding the lower limit of the first interval and the upper limit of the last interval) $d_{2k_e} = d_{Qn} + 0.5\Delta = k_e\Delta + \hat{d}_n + 0.5\Delta$ and $d_{1k_e} = d_{Qn} - 0.5\Delta = k_e\Delta + \hat{d}_n - 0.5\Delta$.

As we obtain $e_{Q_n}$ from $e_n$ by saturation, the probability of occurrence of values beyond the quantization interval must be added to the probability of occurrence of the extreme values in the $e_{Q_n}$ distribution. Then, the pdf of $e_{Q_n}$ is given by:

$$\begin{aligned} &f_{\mathbf{e_Q}}(e_Q|k\Delta, w_n) \\ &= \sum_{k_1=-2^{(b-1)}}^{k_1=2^{(b-1)}-1} Pr\{e_Q = k_1\Delta|k\Delta, w_n\}\delta(e_Q - k_1\Delta) \end{aligned} \tag{11}$$

where

$$\begin{aligned} &\Pr\{e_Q = k_1\Delta|k\Delta, w_n\} \\ &= \begin{cases} \Pr\{e = k_1\Delta|k\Delta, w_n\}, & -1 < e_{Qn} < 1 \\ \displaystyle\sum_{k_e=2^{(b-1)}-1}^{k_e=2^b-1} \Pr\{e = k_e\Delta|k\Delta, w_n\}, & e_{Qn} = 1-\Delta \\ \displaystyle\sum_{k_e=-2^b+1}^{k_e=-2^{(b-1)}} \Pr\{e = k_e\Delta|k\Delta, w_n\}, & e_{Qn} = -1 \end{cases} \end{aligned} \tag{12}$$

### 3.3. Conditional pdf of $y_{Qn}$

Consider $y_n = \mu e_{Qn} x_n$, with $e_{Qn}$ and $x_n$ quantized values. Then,

$$y_n = \mu k_e \Delta k \Delta = \mu(k_e k)\Delta^2 = \mu k_y \Delta_y \tag{13}$$

where $k_y = k_e\, k$ and $\Delta_y = \Delta^2$.

Since $\max\{k_y\}$ occurs for $\min\{k_e\}$ and $\min\{k\}$, then $k_{y_{max}} = 2^{2(b-1)}$. Now, $\min\{k_y\}$ occurs for $\min\{k_e\}$ and $\max\{k\}$ or vice-versa. Then, $k_{y_{min}} = -2^{(b-2)}(2^b - 2)$. Since $y_n = \mu e_{Qn} x_n$, we have that $e_{Qn} = y_n/(\mu x_n) = k_y\Delta/k$. Then,

$$\begin{aligned} \Pr\{y_n = y|k\Delta, w_n\} &= \Pr\{\mu e_{Qn} x_n = y|k\Delta, w_n\} \\ &= \Pr\left\{ e_{Qn} = \frac{k_y\Delta}{k} \middle| k\Delta, w_n \right\} \end{aligned} \tag{14}$$

where $\Pr\{e_Q|k\Delta, w_n\}$ can be determined from (11) and (12). Thus, the pdf of $y_n$ can be determined from (14) for $k_y$ running from $k_{y_{min}}$ to $k_{y_{max}}$. However, as $y_n$ results from the multiplication of $b$-bit numbers, its values will not be integer multiples of $\Delta$. Quantizing $y_n$ to $b$ bits yields $y_{Q_n}$, whose pdf is then given by

$$\begin{aligned} &f_{\mathbf{y_Q}}\{y_Q|k\Delta, w_n\} \\ &= \sum_{k_{y_Q}=-2^{(b-1)}}^{k_{y_Q}=2^{(b-1)}-1} \Pr\{y_Q = k_{y_Q}\Delta|k\Delta, w_n\}\,\delta(y_Q - k_{y_Q}\Delta) \end{aligned} \tag{15}$$

with

$$\begin{aligned} &\Pr\{y_Q = k_{y_Q}\Delta|k\Delta, w_n\} \\ &= \Pr\{y \geq y_1|k\Delta\}\mathrm{U}(y-y_1) - \Pr\{y \geq y_2|k\Delta\}\mathrm{U}(y-y_2) \end{aligned} \tag{16}$$

where $y_1 = k_{y_Q}\Delta - 0.5\Delta$, $y_2 = k_{y_Q}\Delta + 0.5\Delta$ and $\mathrm{U}(\cdot)$ is the unit step function.

Having determined the statistics of $y_{Q_n}$, we proceed to determine the conditional pdf of $w_{n+1}$.

### 3.4. Conditional pdf of $w_{n+1}$

To simplify the expressions, we define $\omega_{n+1} = w_n + y_{Qn}$. Then, $\omega_{n+1} = w_n + Q\{\mu e_{Qn} x_n\} = w_n + Q\{y_n\}$ and

$$\begin{aligned} &\Pr\{\omega_{n+1} = k_{w1}\Delta|k\Delta, w_n\} = \Pr\{w_n + y_Q = k_{w1}\Delta|k\Delta, w_n\} \\ &= \Pr\{y_Q = k_{w1}\Delta - w_n|k\Delta, w_n\} \end{aligned} \tag{17}$$

This probability can be determined from (15) and (16). Nevertheless, to determine the statistics of $w_{n+1}$ we must include the saturation effects in (17). Then,

$$\begin{aligned} &f_{\mathbf{w}}\{w_{n+1}|k\Delta, w_n\} \\ &= \sum_{k_w=-2^{b-1}}^{k_w=2^{b-1}-1} \Pr\{w_{n+1} = k_w\Delta|k\Delta, w_n\}\delta(w_{n+1} - k_w\Delta) \end{aligned} \tag{18}$$

where

$$\begin{aligned} &\Pr\{w_{n+1} = k_w\Delta|k\Delta, w_n\} \\ &= \begin{cases} \Pr\{\omega_{n+1} = k_w\Delta|k\Delta, w_n\}, & -1 < w_{n+1} < 1 \\ \displaystyle\sum_{k_{w1}=2^{(b-1)}-1}^{k_{w1}=2^b-2} \Pr\{\omega_{n+1} = k_{w1}\Delta|k\Delta, w_n\}, & w_{n+1} = 1-\Delta \\ \displaystyle\sum_{k_{w1}=-2^b}^{k_{w1}=-2^{(b-1)}} \Pr\{\omega_{n+1} = k_{w1}\Delta|k\Delta, w_n\}, & w_{n+1} = -1 \end{cases} \end{aligned} \tag{19}$$

Eq. (19) allows the determination of the matrix of transition probabilities $P_w$ for the Markov chain $w_n$. This matrix provides information about the dynamics of the weights and about the convergence of the mean-square estimation error (MSE).

## 4. CONDITION ON THE STEP SIZE

Once we have determined $P_w$ for the Markov chain, we can determine conditions to be satisfied so that $P_w$ has desirable properties, which translate into an adequate behavior of the LMS algorithm in finite precision. For instance, if $w_n$ is to converge to a unique optimum, $P_w$ matrix must corresponds to an aperiodic and, at least, a semi-ergodic Markov chain [10].

A typical problem with quantized adaptive algorithms is the possibility of premature stop or drastic slow-down [8]. To avoid this phenomenon, we must guarantee that $|y_{Q_n}| = |Q\{\ \mu e_{Qn} x_n\ \}| \geq \Delta$ or, equivalently, that $|\mu e_{Qn} x_n| \geq \Delta/2$. Considering a high signal-to-noise ratio (SNR) (the worst case), $|e_{Q_n}| \geq \Delta$ whenever $w_n \neq \hat{w}$. Then the above condition requires $|\mu x_n| \geq \frac{1}{2}$. The minimum value of $\mu$ that satisfies this condition occur for $|x_n| = 1$. Thus, the condition on $\mu$ to avoid stopping is

$$\mu \geq \frac{1}{2} \tag{20}$$

We use this limit in the following example.

## 5. PERFORMANCE ESTIMATION

In this section we illustrate the use of the theoretical model to determine the finite precision LMS adaptive filter behavior.

Consider an example with SNR $= 20dB$, $\mu = 1$ (thus, $\mu \geq 1/2$), $b = 3$ bits (8 quantization levels), $\hat{w} = 0.75$, $\sigma_z^2 = 0.0034$ and $w_0 = -1$. Using (18) and (19), we determine the transition probabilities for $w_n$. The matrix $P_w^\infty$ (steady-state) is given by:

$$w_n/w_{n+1} \rightarrow \quad \begin{bmatrix} -1 & -0.75 & -0.5 & -0.25 & 0.00 & 0.25 & 0.50 & 0.75 \end{bmatrix}$$
$$\downarrow$$

$$\begin{bmatrix} -1.00 \\ -0.75 \\ -0.50 \\ -0.25 \\ 0.00 \\ 0.25 \\ 0.5 \\ 0.75 \end{bmatrix} P_w^\infty = \begin{bmatrix} 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.31 & 0.69 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.31 & 0.69 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.31 & 0.69 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.31 & 0.69 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.31 & 0.69 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.31 & 0.69 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.31 & 0.69 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.31 & 0.69 \end{bmatrix}$$

This matrix corresponds to a semi-ergodic Markov chain [10]. It shows that $E[w_n] = 0.5 \times 0.31 + 0.75 \times 0.69 = 0.6725$ in steady-state. The steady-state coefficient misadjustment is then $-0.1\%$. Using (11) and (12), it is possible to determine the probability of the quantized error $e_{Q_n}$ for each state of $w_n$. Therefore, we can evaluate the MSE at the $n$-th iteration from the knowledge of $P_w^n$. Table 1 shows the MSE at iteration $n$ for each possible value of $w_n$.

**Table 1**. MSE for each state of $w_n$

| $w_n$ | MSE |
|---|---|
| $-1$ | 0.5446 |
| $-0.75$ | 0.5445 |
| $-0.5$ | 0.4781 |
| $-0.25$ | 0.3876 |
| 0 | 0.1966 |
| 0.25 | 0.0751 |
| 0.5 | 0.0204 |
| 0.75 | 0.0126 |

Then, to determine the MSE at iteration $n$ we multiply the MSE vector by the row of $P_w^n$ corresponding to the initial weight value $w_0$. For this example $w_0 = -1$. Thus, the first line of $P_w^n$ must be considered. In steady-state, we multiply the first row of $P_w^\infty$ by the MSE vector in Table 1, yielding $\lim_{n\rightarrow\infty} E[e_n^2] = 0.015$ or -18.24dB. Fig. 3 shows the MSE obtained from Monte Carlo simulations (100 runs) as the smooth line. The dots correspond to estimations evaluated using the theoretical model. The matrices $P_w^n$ used for the transient period are not shown for space limitations. Note that there is excellent agreement between simulation and theory.

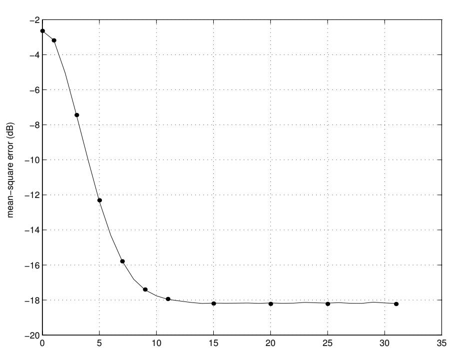


**Fig. 3**. Monte Carlo simulations of eq.(1) (-) versus the MSE obtained from theory ($\cdot\cdot$).

## 6. CONCLUSIONS

This paper presented a new model for the behavior of the LMS algorithm when implemented in finite precision. The adaptive filter coefficients were modeled as a Markov chain and the matrix of transition probabilities of the chain was determined for the one-dimensional case. Linearizations used in other models available in the literature are eliminated. The new model includes saturation effects in the quantization process. The theoretical results can be used to determine the transient and steady-state behaviors of the adaptive weights and MSE. Monte Carlo simulation results illustrate the quality of the proposed model.


## 7. ACKNOWLEDGMENTS

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