CONVEX COMBINATION OF BLIND ADAPTIVE EQUALIZERS WITH DIFFERENT TRACKING CAPABILITIES

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ABSTRACT

We propose a convex combination of two blind equalizers adapted respectively by CMA (Constant Modulus Algorithm) and SWA (Shalvi-Weinstien Algorithm). The performance of the proposed scheme is, in the worst case, as good as that of the best of its components. This behavior provides a good tracking capability, since both CMA or SWA may have a better tracking performance, depending on the kind of nonstationary environment. A steady-state analysis (using energy conservation) is also presented, considering both the proposed scheme, and the convex combination of two CMAs.

Index Terms— Adaptive filters, blind equalization, convex combination, energy conservation, tracking analysis, Constant Modulus Algorithm.

1. INTRODUCTION

In order to improve adaptive filter performance, some attention has been given to the combination of algorithms. A convex combination of one fast and one slow LMS (Least Mean-Square) filter was introduced in [1] and analyzed via energy conservation relations in [2]. Furthermore, it was shown that this structure is universal in the mean-square error sense, presenting a worst case performance as good as that of the best of its components, and outperforming both of them when the correlation between the a priori errors of the component filters is low enough [2]. The scheme was also proposed for blind equalization applications, considering the combination of one fast and one slow CMA (Constant Modulus Algorithm) [3], but without a theoretical model for the overall mean-square error.

Another well-known blind equalization algorithm is the Shalvi-Weinstein Algorithm (SWA) [4]. Based on the link between blind equalization and classical adaptive filtering of [5], CMA and SWA can be interpreted as the blind versions of LMS and RLS (Recursive Least-Squares), respectively. Thus, as shown in [6], the ratio between the minimum EMSE (Excess Minimum Square Error) for SWA and CMA is the same as between the RLS and LMS algorithms, in nonstationary environments. Consequently, as with LMS and RLS, the tracking behavior of CMA and SWA depends on the kind of nonstationary environment [7, 8]. They present similar performances when the matrix $\mathbf{Q}$ of the random-walk model in (7) is a multiple of the identity matrix. For other choices of $\mathbf{Q}$, one algorithm may perform better than the other. Taken together, the results of [5–7] mean that, if $\mathbf{Q}$ is a multiple of $\mathbf{R}$ (autocorrelation of the input signal), CMA is superior, and if $\mathbf{Q}$ is a multiple of $\mathbf{R}^{-1}$, SWA is superior.

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This paper has two main contributions. First, to take advantage of the differences in the tracking performances of CMA and SWA [6], employing a convex combination of them. The proposed combination should acquire the good initial convergence properties of SWA, and have a good tracking performance for all nonstationary environments. Second, using energy conservation relations, we present a tracking analysis for the proposed scheme, and also for the convex combination of two CMAs proposed in [3]. In both cases, we consider nonconstant modulus, real-valued constellations (the extension to complex constellations is straightforward).

The paper is organized as follows. In the next section, we describe convex combinations of blind equalizers. In Section 3, the steady-state analysis is presented. Then, simulation results and the conclusions are shown in sections 4 and 5, respectively.

2. PROBLEM FORMULATION

A communication system model considering a T/L fractionally-spaced equalizer is shown in Figure 1.

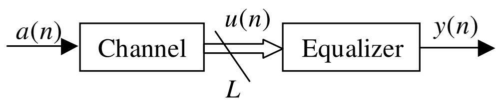

Fig. 1. Communication system model considering a T/L fractionally-spaced equalizer.

Under certain well-known conditions, this model ensures perfect equalization in a noise-free environment, e.g. [9, 10]. The transmitted signal $a(n)$ is assumed i.i.d. (independent and identically distributed) and non Gaussian. We assume M-tap FIR equalizers, with input vector $\mathbf{u}(n)$. The output of an equalizer, in a noise free environment, can be written as

$$y(n) = \mathbf{u}^T(n)\mathbf{w}(n-1),$$

where $\mathbf{w}(n-1)$ is the equalizer weight vector. The blind equalizer must mitigate the channel effects without training data and recover the signal $a(n)$ for some delay τ_d.

A convex combination of two equalizers can also be used, as proposed in [3] and depicted in Figure 2. In this scheme, the overall weight vector is given by $\mathbf{w}(n-1) = \eta(n)\mathbf{w}_1(n-1) + [1 - \eta(n)]\mathbf{w}_2(n-1)$. Similarly, the output of the overall equalizer is given by

$$y(n) = \eta(n)y_1(n) + [1-\eta(n)]y_2(n), \qquad (1)$$

where $y_i(n)$, $i = 1, 2$ are the outputs of the equalizers, i.e., $y_i(n) = \mathbf{u}^T(n)\mathbf{w}_i(n-1)$, $\mathbf{u}(n)$ is the common regressor vector, and $\mathbf{w}_i(n-1)$ are the weight vectors of each component equalizer. The mixing

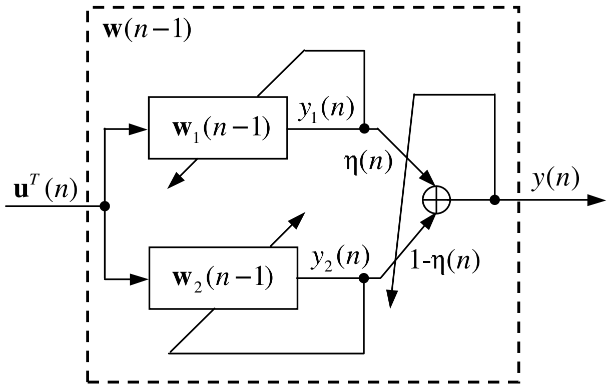

Fig. 2. Adaptive convex combination of two blind equalizers.

parameter $\eta(n)$ is modified via a sigmoidal function and the auxiliary variable $\alpha(n-1)$ [1–3], that is,

$$\eta(n) = \text{sgm}[\alpha(n-1)] = \frac{1}{1+e^{-\alpha(n-1)}}, \tag{2}$$

with $\alpha(n)$ being updated as

$$\alpha(n) = \alpha(n-1) - \mu_\alpha e_\alpha(n)\eta(n)[1-\eta(n)], \tag{3}$$

where

$$e_\alpha(n) = (y^2(n) - r^a)y(n)[y_1(n) - y_2(n)], \tag{4}$$

$r^a = \text{E}\{a^4(n)\}/\text{E}\{a^2(n)\}$, and $\text{E}\{\cdot\}$ denotes the expectation operation. Equation (3) was obtained in [3], using a stochastic gradient rule to minimize the instantaneous constant modulus cost given by $J(n) = (y^2(n) - r^a)^2$. The auxiliary variable $\alpha(n)$ is used to keep $\eta(n)$ in the interval $[0, 1]$. A drawback of this scheme is that $\alpha(n)$ stops updating whenever $\eta(n)$ is close to 0 or 1. To avoid this, [2, 3] suggest that $\alpha(n)$ be restricted (by simple saturation) to lie inside a symmetric interval $[-\alpha^+,\ \alpha^+]$.

We generalize the above results, assuming that the coefficients $\mathbf{w}_i$, $i = 1, 2$ are adapted by algorithms of the form

$$\mathbf{w}_i(n) = \mathbf{w}_i(n-1) - \rho_i\,\mathbf{M}_i(n)\mathbf{u}(n)e_i(n), \tag{5}$$

where

$$e_i(n) = (y_i^2(n) - r^a)y_i(n), \tag{6}$$

ρ_i is a step-size, and $\mathbf{M}_i(n)$ is a symmetric and non-singular matrix. CMA and SWA employ the step-sizes and the matrices $\mathbf{M}_i(n)$ as in Table 1. In this table, $\mathbf{I}$ is the identity matrix, $0 \ll \lambda_i < 1$ is the forgetting factor, $\gamma \triangleq 3\text{E}\{a^2(n)\} - r^a$. For SWA, the matrix $\mathbf{M}_i(n)$ is obtained by using the matrix inversion lemma [8, p.67]. $\widehat{\mathbf{R}}(n)$ is an estimate of the autocorrelation matrix of the input signal $\mathbf{R} \triangleq \text{E}\{\mathbf{u}(n)\mathbf{u}^T(n)\}$.

3. TRACKING ANALYSIS

In a nonstationary environment, the variation in the zero-forcing solution $\mathbf{w}_\text{o}$ is assumed to follow the random-walk model [8, p. 359], that is,

$$\mathbf{w}_\text{o}(n) = \mathbf{w}_\text{o}(n-1) + \mathbf{q}(n). \tag{7}$$

In this model, $\mathbf{q}(n)$ is an i.i.d. sequence with positive definite autocorrelation matrix $\mathbf{Q} = \text{E}\{\mathbf{q}(n)\mathbf{q}^T(n)\}$ and is independent of the

Table 1. Parameters of CMA and SWA.

Alg.	ρ_i	$\mathbf{M}_i^{-1}(n)$
CMA	μ_i	$\mathbf{I}$
SWA	γ^{-1}	$\widehat{\mathbf{R}}(n) = \sum_{l=1}^{n} \lambda_i^{n-l}\mathbf{u}(l)\mathbf{u}^T(l)$

initial conditions $\{\mathbf{w}_\text{o}(-1), \mathbf{w}(-1)\}$ and of $\{\mathbf{u}(l)\}$ for all $l < n$ [8, Sec. 7.4].

Let the overall *a priori* error be $e_a(n) = \mathbf{u}^T(n)\widetilde{\mathbf{w}}(n-1)$, where $\widetilde{\mathbf{w}}(n-1) = \mathbf{w}_\text{o}(n-1) - \mathbf{w}(n-1)$. One measure of equalizer performance is given by the steady-state EMSE, defined as $\zeta \triangleq \lim_{n\to\infty} \text{E}\{e_a^2(n)\}$. The *a priori* error $e_a(n)$ can be written as a function of the *a priori* errors of the component equalizers, i.e.,

$$e_a(n) = \eta(n)e_{a,1}(n) + [1-\eta(n)]e_{a,2}(n), \tag{8}$$

where $e_{a,i}(n) = \mathbf{u}^T(n)\widetilde{\mathbf{w}}_i(n-1)$ and $\widetilde{\mathbf{w}}_i(n-1) = \mathbf{w}_\text{o}(n-1) - \mathbf{w}_i(n-1),\ i = 1, 2$.

Using the energy conservation approach of [8, Ch. 7], a steady-state analysis of algorithms of the form (5) was presented in [6, 11, 12]. The steady-state EMSE of CMA, in a nonstationary environment, is given by [12, Table III]

$$\zeta^{\text{CMA}} \approx \frac{\mu\xi\text{Tr}(\mathbf{R})}{2\gamma} + \frac{\text{Tr}(\mathbf{Q})}{2\gamma\mu}, \tag{9}$$

where $\xi \triangleq \text{E}\{a^6(n) - (r^a)^2a^2(n)\}$, $\text{Tr}(\mathbf{A})$ stands for the trace of matrix $\mathbf{A}$. For SWA, we have [6, Table II]

$$\zeta^{\text{SWA}} \approx \frac{M(1-\lambda)\xi}{2\gamma^2} + \frac{\text{Tr}(\mathbf{QR})}{2(1-\lambda)}. \tag{10}$$

Using the same arguments of [2, Sec. III], it is possible to show that the considered convex combination is universal in the mean square error sense. Thus, considering the convex combination of SWA and CMA, when SWA outperforms CMA in the steady-state, the behavior of the overall filter will be close to that of SWA and $\zeta \approx \zeta^{\text{SWA}}$. On the other hand, when CMA is superior, $\zeta \approx \zeta^{\text{CMA}}$. Moreover, there are situations where the combination will outperform both of them. In this case, the EMSE of the overall filter will be close to

$$\zeta \approx \zeta_{12} + \frac{\Delta\zeta_1\Delta\zeta_2}{\Delta\zeta_1 + \Delta\zeta_2}, \tag{11}$$

where ζ_{12} is the cross-EMSE , defined as

$$\zeta_{12} \triangleq \lim_{n\to\infty} \text{E}\{e_{a,1}(n)e_{a,2}(n)\}, \tag{12}$$

and $\Delta\zeta_i = \zeta_i - \zeta_{12},\ i = 1, 2$. Eq. (11) was obtained in [2, Eq. (33)] for the combination of two LMS filters. However, it is also valid for the convex combination of blind algorithms of the form (5), as for example, for the combination of two CMAs, one CMA and one SWA, or two SWAs. The EMSE of the overall filter is the minimum of the values calculated by the expressions of each component filter and (11).

Analytical expressions for ζ_{12} for blind algorithms have not been computed before. In the following analysis, we obtain such expressions for the combinations of two CMAs with different step-sizes (μ_1-CMA and μ_2-CMA), and one SWA with one CMA (λ_1-SWA with μ_2-CMA). Subtracting both sides of (5) from $\mathbf{w}_o(n)$ and using (7) for $i = 1, 2$, we arrive at

$$\widetilde{\mathbf{w}}_i(n) - \mathbf{q}(n) = \widetilde{\mathbf{w}}_i(n-1) + \rho_i \mathbf{M}_i(n)\mathbf{u}(n)e_i(n). \tag{13}$$

In order to obtain ζ_{12}, we multiply the transpose of (13) with $i = 2$ by (13) with $i = 1$ (for the combination of one SWA with one CMA, we use $\widehat{\mathbf{R}}(n)$ as a weighting matrix), and take expectations. To simplify the resulting expression, the following assumptions are considered:

A1. $\mathrm{E}\{a(n)\} = 0, \gamma > 0$ [4,8].

A2. When $n \to \infty$, $\mathrm{E}\{\widetilde{\mathbf{w}}_2^T(n)\mathbf{A}\widetilde{\mathbf{w}}_1(n)\} = \mathrm{E}\{\widetilde{\mathbf{w}}_2^T(n-1)\mathbf{A}\widetilde{\mathbf{w}}_1(n-1)\}$ for any weighting matrix $\mathbf{A}$. This assumes that the filters are operating in stable conditions, and have reached steady-state.

A3. $a(n-\tau_d)$ and $e_{a,i}(n)$, $i = 1, 2$ are independent at the steady-state. This assumption was used in the steady-state analysis of CMA [11, *As.I.1*,p.84] and essentially requires the estimation error $\{e_{a,i}(n)\}$ of the equalizer to be insensitive, at steady-state, to the actual transmitted symbols $\{a(n)\}$.

A4. The squared Euclidean norm of the regressor $\mathbf{u}(n)$, i.e., $\|\mathbf{u}(n)\|^2$, and $e_i(n)$, $i = 1, 2$ are independent at the steady-state. This requires the energy of the input vector to be independent of the equalizer output [11, *As.I.2*, p.84].

A5. Eq. (6) can be rewritten using the approximation $y_i(n) \approx a(n - \tau_d) - e_{a,i}(n)$ since at the steady-state $a(n - \tau_d) \approx \mathbf{u}^T(n)\mathbf{w}_o(n-1)$.

A6. $\lim_{n\to\infty} \mathrm{E}\{e_1(n)e_2(n)\} \approx \xi$. To obtain this approximation, we use A1, A3, and A5, and consider that terms depending on higher-order combinations of $e_{a,1}(n)$ and $e_{a,2}(n)$ can be disregarded. This means that we assume
$$|\mathrm{E}\{e_{a,1}^{\ell}(n)e_{a,2}^{k}(n)\}| \ll |\xi|, \quad \ell + k \geq 2.$$
Similar assumptions were employed in [11, *Th.3* and *Th.4*].

A7. $\mathrm{E}\{e_1(n)e_{a,2}(n)\} = \mathrm{E}\{e_2(n)e_{a,1}(n)\} \approx -\gamma\zeta_{12}$, in steady-state. This assumption is obtained by using A3, A5, and considering that, for $\ell = 1$ and $k = 3$, or $\ell = 3$ and $k = 1$, $\gamma|\zeta_{12}| \gg |\mathrm{E}\{e_{a,2}^{\ell}(n)e_{a,1}^{k}(n)\}|$.

A8. In the analysis of one SWA and one CMA, using $\widehat{\mathbf{R}}(n)$ as weighting matrix, we obtain

$$\mu \mathrm{E}\{e_2(n)\mathbf{u}^T(n)\widehat{\mathbf{R}}(n)\widetilde{\mathbf{w}}_1(n-1)\} \approx \frac{-\gamma\mu \mathrm{Tr}(\mathbf{R})}{(1-\lambda)M}\zeta_{12}. \tag{14}$$

To obtain this approximation, we assume that the regressor $\mathbf{u}(n)$ follows the model proposed in [13]. The idea is to assume a simple model for $\mathbf{u}(n)$, but in such a way that the autocorrelation $\mathbf{R}$ is preserved. We thus assume that $\mathbf{u}(n)$ may point to any of the directions of one of the eigenvectors of $\mathbf{R}$ with equal probability, but with amplitude proportional to the square-root of the corresponding eigenvalue ϱ_i. Using this model, (14) follows from the fact that $\mathbf{u}^T(n)\mathbf{R} = \varrho_i \mathbf{u}^T(n)$ for some i at every instant.

Considering now the random-walk model for $\mathbf{q}(n)$ and Assumption A2, we get for the considered combinations

$$\begin{aligned} -\mathrm{E}\{\mathbf{q}^T(n)\mathbf{A}\mathbf{q}(n)\} &\approx \rho_1 \mathrm{E}\{e_{a,2}(n)e_1(n)\} + \\ &+ \rho_2 \mathrm{E}\{e_2(n)\mathbf{u}^T(n)\mathbf{A}\widetilde{\mathbf{w}}_1(n-1)\} + \\ &+ \rho_1\rho_2 \mathrm{E}\{e_1(n)e_2(n)\mathbf{u}^T(n)\mathbf{u}(n)\}. \end{aligned} \tag{15}$$

Using A1, A3–A7, after some algebra in (15), for the combination of μ_1-CMA and μ_2-CMA, with $\mathbf{A} = \mathbf{I}$, we get

$$\zeta_{12} \approx \frac{\mu_1\mu_2 \mathrm{Tr}(\mathbf{R})\xi + \mathrm{Tr}(\mathbf{Q})}{\gamma(\mu_1 + \mu_2)}. \tag{16}$$

Analogously, for the combination of λ_1-SWA and μ_2-CMA, with $\mathbf{A} = \widehat{\mathbf{R}}(n)$ and A1, A3–A8, we arrive at

$$\zeta_{12} \approx \frac{\dfrac{\mu_2 \mathrm{Tr}(\mathbf{R})\xi}{\gamma} + \dfrac{\mathrm{Tr}(\mathbf{QR})}{1-\lambda_1}}{1 + \mu_2\gamma\dfrac{\mathrm{Tr}(\mathbf{R})}{M(1-\lambda_1)}}. \tag{17}$$

In stationary environments, the expressions can be simplified, making $\mathbf{Q} = \mathbf{0}$.

4. SIMULATION RESULTS

To verify the behavior of the proposed convex combination and the validity of the tracking analysis, we assume $\mathbf{Q} = \beta^2\mathbf{R}$ or $\mathbf{Q} = \beta^2\mathbf{R}^{-1}$ with $\beta = 0.001$, 6-PAM with statistics $\mathrm{E}\{a^6(n)\} = 5451.7$, $\mathrm{E}\{a^2(n)\} = 11.67$, and $r^a = 20.2$, and channel coefficients $[0.1,\ 0.3,\ 1,\ -0.1,\ 0.5,\ 0.2]$ [11]. In the combinations, each component filter has $M = 4$ coefficients as a $T/2$-fractionally spaced equalizer and is initialized with only one non-null element in the second position.

Figure 3-a) shows the EMSE estimated from the ensemble-average of 200 independent runs for μ_1-CMA, μ_2-CMA, and their convex combination, with $\mu_1 = 10^{-4}$, $\mu_2 = 5 \times 10^{-5}$, $\mu_\alpha = 0.1$, and $\alpha^+ = 4$. To facilitate the visualization, the curves were filtered by a moving-average filter with 64 coefficients. At iteration $n = 45000$, matrix $\mathbf{Q}$ is changed from $\mathbf{Q} = \beta^2\mathbf{R}$ to $\mathbf{Q} = \beta^2\mathbf{R}^{-1}$, with $\beta = 0.001$. As μ_1-CMA presents faster convergence than μ_2-CMA, the combination performs close to μ_1-CMA during the first 4000 iterations. At steady-state, when μ_2-CMA outperforms μ_1-CMA, the combined scheme presents a performance close to that of μ_2-CMA, independently of the nonstationary environment. In Figure 3-b), we show $\mathrm{E}\{\eta(n)\}$, confirming this observation. The dashed lines in the figure show the predicted values of ζ for each algorithm and their combination. Although there is no an exact agreement between analysis and simulation, the predicted values model the overall behavior of the algorithms. Note that a difference of a few dB is common in models for blind algorithms, due to the strong assumptions necessary for the analysis.

For the same conditions, we show in Figure 4, the behavior of λ_1-SWA ($\lambda_1 = 0.999$), μ_2-CMA and their convex combination. In this case, when $\mathbf{Q} = \beta^2\mathbf{R}$, the proposed scheme performs close to μ_2-CMA, being slightly better than it at the steady-state. However, its convergence is close to that of λ_1-SWA, during the first 2000 iterations. When $\mathbf{Q} = \beta^2\mathbf{R}^{-1}$, λ_1-SWA outperforms μ_2-CMA, and the combination switches back to SWA. Again, $\mathrm{E}\{\eta(n)\}$, shown in Figure 4-b), confirms the behavior of the combined structure. We should notice that the EMSE of SWA predicted by (10) for $\mathbf{Q} = \beta^2\mathbf{R}$ shows the worst disagreement ($\approx$ 2 dB), which is not very large for the current application. Comparing the two combinations, we observe that λ_1-SWA with μ_2-CMA can be a better alternative than μ_1-CMA with μ_2-CMA, in regards of tracking performance.

5. CONCLUSIONS

We propose a convex combination of the CMA and SWA algorithms to take advantage of their different tracking behavior. Through simulations, we verified that the scheme is universal, performing as the

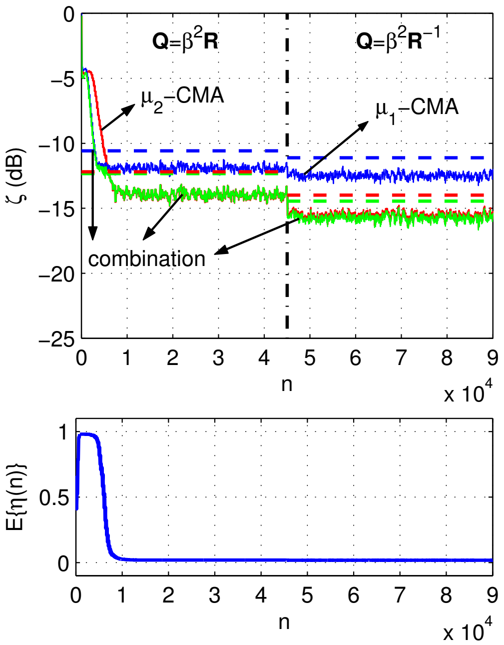

Fig. 3. a) EMSE for μ_1-CMA, μ_2-CMA, and their convex combination and b) ensemble-average of $\eta(n)$; $\mu_1 = 10^{-4}$, $\mu_2 = 5 \times 10^{-5}$, $\mu_\alpha = 0.1$, $\alpha^+ = 4$, $\beta = 0.001$; mean of 200 independent runs. In a), the dashed lines represent the predicted values of ζ for each algorithm and their combination.

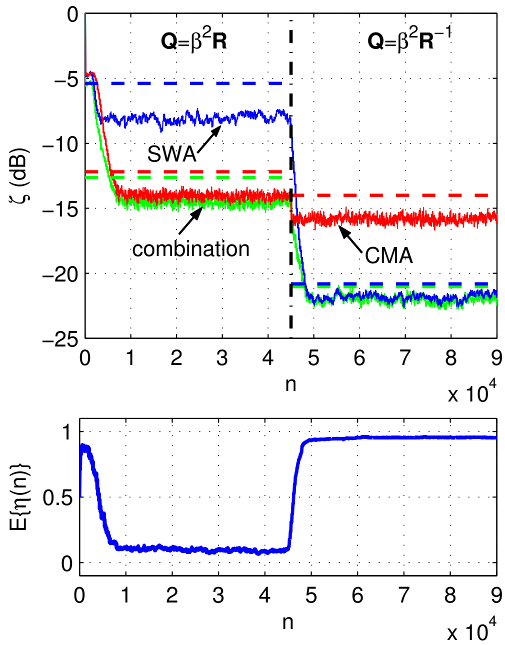

Fig. 4. a) EMSE for λ_1-SWA, μ_2-CMA, and their convex combination and b) ensemble-average of $\eta(n)$; $\lambda_1 = 0.999$, $\mu_2 = 5 \times 10^{-5}$, $\mu_\alpha = 0.1$, $\alpha^+ = 4$, $\beta = 0.001$; mean of 200 independent runs. In a), the dashed lines represent the predicted values of ζ for each algorithm and their combination.

best component filter and being better than both of them in some situations. Using energy conservation relations, we obtained analytical expressions for the steady-state cross-EMSE of the combinations of two CMA and CMA with SWA. Using these expressions, we estimate the steady-state EMSE of the overall equalizer and find it in reasonable agreement with experimental results.

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