# The Karhunen-Loève Transform of Discrete MVL Functions*

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## Abstract

*The Karhunen-Loève (KL) transform of a discrete multiple-valued logic function is studied with respect to algebraic graph theory. The spectrum of a Cayley graph defined over the symmetry group is observed to be equivalent to the KL spectrum of a discrete function when the Cayley graph is generated using that function. It is also observed that the autocorrelation of the discrete function using the symmetry group operator is equivalent to the adjacency matrix of the Cayley graph. In addition to the theoretical interests, the KL spectrum of a discrete multiple-valued logic function can have applications in compact function representation and the determination of function estimates with a reduced support set. Example computations are shown in addition to the presentation of the mathematical properties.*

## 1 Introduction

The KL transform is named for Kari Karhunen and Michel Loève who developed it as a series expansion method for continuous random processes [6, 10, 11]. Originally, Harold Hotelling studied the discrete formulation of the KL transform [4] and for this reason, the Karhunen-Loève (KL) transform is also known as the Hotelling transform. The KL transform is heavily utilized for performance evaluation of compression algorithms in the digital signal processing community since it has been proven to be the optimum transform for the compression of a sampled sequence in the sense that the KL spectrum contains the largest number of zero-valued coefficients [5]. Because the basis functions of the KL transform are data dependent, the KL spectrum is generally used as a benchmark to judge the effectiveness of the data compression capability of other more easily computed transforms.

The KL transform is also used in clustering analysis to determine a new coordinate system for sample data where the largest variance of a projection of the data lies on the first axis, the next largest variance on the second axis, and so on. Because these axes are orthogonal, this approach allows for reducing the dimensionality of the data set by eliminating those coordinate axes with small variances. This data reduction technique is commonly referred to as *Principle Component Analysis* (PCA) [3] by the analysts who employ it for this purpose. An example of the recent application of PCA in a computer vision system is [14].

The KL transform is a unitary transform with basis functions that are orthogonal eigenvectors of the covariance matrix of a data set or measurement vector. In this paper, we are interested in discrete *Multiple-valued Logic* (MVL) functions, thus the eigenvectors of the autocorrelation matrix of the function form the basis. The KL spectrum is defined as the set of eigenvalues associated with the basis functions. It is the fact that the basis functions depend on the actual function to be transformed that discourages widespread use of the KL transform as compared to other transforms that have a common, well-known set of basis functions independent of the function to be transformed and often with other desirable features such as a recursive construction characteristic.

There has been interest in the relationship between algebraic graph theory and the spectra of discrete functions. This relationship can be conveniently described using particular algebraic groups and generators that correspond to Cayley graphs. In particular, it has been shown that the Walsh spectrum of a binary-valued function $f(x_1, x_2, \ldots, x_n)$ may be computed as the spectrum of a Cayley graph

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over the elementary additive Abelian group $\mathbf{Z_{2^n}}$ using a generator based on $f$ [1]. These results were generalized to provide a technique to compute the Chrestenson spectrum of finite non-binary discrete-valued functions in [15] and extended to handle the case of mixed-radix functions in [16]. Here, we investigate the extension of these previous techniques using the non-Abelian symmetry groups instead of the Abelian additive (mod(p)) groups. Interestingly, it is observed that this approach yields the KL transform of the function of interest when the symmetric group product operator is used to compute the autocorrelation matrix. We note that this is a specific non-Abelian group as in contrast to the more general work where Fourier transforms over non-Abelian groups are studied as described in [8, 12]. This work focuses upon the inter-relationship between autocorrelation functions, Cayley graph spectra, and the KL spectrum of a discrete function.

The KL transform of a discrete MVL function can have potential application in serving as a benchmark in the evaluation of methods to represent compact forms of the function. An example may be the evaluation of various decision diagram structures that are formulated to take advantage of zero-valued terminal vertices or as compact signatures in large cell libraries. A related technique is described in [9] where autocorrelation properties are used to find minimal DD representations. Another possible application is to use the KL spectra in a mode similar to PCA to approximate a function of $n$ variables with a function of $m$ variables where $n > m$. This could be accomplished by establishing a heuristic that yields a threshold for the variance values below which to discard the associated support variables. Finally, there is some merit in evaluating the theoretical relationship between algebraic graph theory and the optimal KL transformation of a discrete MVL function.

The remainder of the paper is organized as follows. First, a section that defines the mathematical definitions of the symmetry group and autocorrelation of a discrete function is provided. This section will also serve to introduce the notation that will be used throughout the following sections. Next, the equivalence of the spectrum of a Cayley graph formed over the symmetric group with a generator depending on a particular function and that functions' corresponding KL spectrum is shown. An example calculation is given and finally conclusions and future work is outlined.

## 2 Symmetry Groups, Cayley Graphs, and Autocorrelation Functions

The symmetric permutation group $\mathbf{S_n}$ can be visualized as consisting of a set of strings with each string consisting of $n$ unique elements and a permutation operation denoted as $\circ$. The number of elements in $\mathbf{S_n}$ is $n!$ since the group product operator $\circ$ results in all possible permutations of the string of length $n$.

### 2.1 Symmetric Permutation Groups and Notation

Symmetric permutation groups consist of elements that can be considered as strings of unique objects. As an example, consider the group $\mathbf{S_3} = (S, \circ)$ where the set of elements $S = \{abc, bca, cab, bac, cba, acb\}$. One way to describe this set of permutations is to use numerical values that correspond to each object (i.e. 1 for $a$, 2 for $b$ and 3 for $c$). The six elements of $S$ may be described with the notation in Table 1. The top row of each element in the table corresponds to $abc$ with the bottom row corresponding to the permuted order. As an example, the element denoted as 1 corresponds to permuting $abc$ to the string $cab$.

**Table 1. Permutation Notation Describing Elements of $\mathrm{S_3}$**

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \equiv 0 \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \equiv 1 \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \equiv 2$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \equiv 3 \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \equiv 4 \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \equiv 5$$

Using the permutation definitions in Table 1, the Cayley table or group operation table corresponding to $\circ$ can be formulated as shown in Table 2. In Table 2 the notation is used as defined in Table 1.

An important characteristic of $\mathbf{S_3}$ is to note that the $\circ$ operation is non-commutative. As an example $2 \circ 5 = 4$; however, $5 \circ 2 = 3$. For this reason $\mathbf{S_3}$ is non-Abelian and it is important to interpret the operations as shown in Table 2 in the proper order. As Table 2 is written here, the entries in the leftmost column correspond to the operand to the left of $\circ$ and the entries at the top of each of column are the operands to the right of $\circ$.

**Table 2. Cayley Table for the $\circ$ Operator over $\mathbf{S_3}$**

| $\circ$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | 2 | 0 | 4 | 5 | 3 |
| 2 | 2 | 0 | 1 | 5 | 3 | 4 |
| 3 | 3 | 5 | 4 | 0 | 2 | 1 |
| 4 | 4 | 3 | 5 | 1 | 0 | 2 |
| 5 | 5 | 4 | 3 | 2 | 1 | 0 |

Also, it is convenient to use the so-called "cyclic notation" to represent a particular permutation. As an example the left-hand side of Equation 1 represents a permutation of a 5-element object using the same notation as that shown in Table 1 and the right-hand side expresses the same permutation in cyclic notation as a single cycle.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 4 & 2 & 5 \end{pmatrix} \quad (1)$$

The interpretation of the cyclic notation is that the first element moves to element 3, element 3 moves to element 4, and so on with the last element (i.e. 5) moving to the first position. Not all permutations can be expressed as a single cycle; however, they can all be expressed as a set of *disjoint cycles*. As an example, consider the permutation of a 5-element object as shown in Equation 2 which is described with two disjoint cycles.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 2 & 5 \end{pmatrix} \quad (2)$$

## 2.2 Cayley Graph Over $\mathbf{S_3}$

A Cayley graph is a graphical representation of an algebraic group. A Cayley graph is denoted as $Cay(V, E)$ where $V$ is the vertex set and $E$ is the edge set. The elements of $V$ correspond uniquely to the the elements of the group. In this case, each $v_i \in V$ corresponds to each element in the set $S$. A subset of $S$, denoted as $Y$ is formed based on the use of one or more generators. A generator operates over a pair in $S$, $(u, v)$ such that some $y_i \in Y$ satisfies $u = y_i v$. When this occurs, an edge $e_i \in E$ is defined as the ordered pair $(u, v)$. In general, a Cayley graph is a directed graph since the edge set is defined as ordered pairs of elements of $V$.

In the case of $\mathbf{S_3}$, $V$ contains vertices corresponding to the elements in $S$. Using the notation in Table 1, the elements of $S$ are defined as integer values $V = \{0, 1, 2, 3, 4, 5\}$. To form $Cay(V, E)$ an appropriate generator, $e(s_i, s_j)$ must be determined. The generator can be considered to be a function that depends on two elements in $(s_i, s_j) \in S$ and yields a color value for the Cayley graph edge connecting the vertices corresponding to $(s_i, s_j)$. If the generator is binary-valued, $Cay(V, E)$ is not necessarily a fully connected graph since edges are not present when $e = 0$. If $e$ is a $p$-valued function, the Cayley graph can be considered to be a fully connected graph with each edge annotated by 1 of $p$ colors. If one of the $p$ values is 0, this can be considered to be the absence of an edge and the resulting Cayley graph is one that is not necessarily complete and contains edges annotated with 1 of $p - 1$ possible colors. In any event, Cayley color graphs can be considered to be a set of non-colored graphs with each graph having an edge present if it has a particular color, otherwise no edge is present.

The spectrum of the Cayley graph is the spectrum of the adjacency matrix representing the graph [2]. The spectrum of a matrix is the set of roots (and their respective multiplicities) of the characteristic matrix commonly referred to as the eigenvalues of the matrix.

## 2.3 Discrete Function Autocorrelation

The autocorrelation of a function gives a measure of that function with itself at all other possible domain values. The autocorrelation of a discrete function as defined in [7] is given in Equation 3.

$$R(u) = \sum_{x=0}^{N-1} f(x) f(x \ominus_p u) \quad (3)$$

Equation 3 assumes that $f$ is a discrete function that depends on a single $p$-valued variable. The operation $\ominus_p$ is the difference modulo-$p$ of the values $x$ and $u$ when $x$ and $u$ are represented in a component-wise manner. We note that this definition of autocorrelation is sometimes referred to as the "one-sided" autocorrelation.

The computation of the autocorrelation of discrete functions can alternatively be expressed in terms of a vector-matrix equation. In this formulation, the discrete autocorrelation function is given as a vector $\mathbf{R_u}$ with each component corresponding to a distinct $u$ value. Likewise the function $f$ is represented as a vector $\mathbf{f}$ where each component is a distinct range value. In this formulation, Equation 4 is used where the $(N-1) \times (N-1)$ matrix $\mathbf{F_u}$ is termed the *autocorrelation matrix*.

$$\mathbf{R_u} = \mathbf{Ff} \tag{4}$$

Using the formulation given in Equation 4, the autocorrelation matrix $\mathbf{F}$ is composed of column vectors as given in Equation 5.

$$\mathbf{F} = [\mathbf{f}(\mathbf{x} \ominus \mathbf{0}) \;\; \mathbf{f}(\mathbf{x} \ominus \mathbf{1}) \; \ldots \; \mathbf{f}\{\mathbf{x} \ominus (\mathbf{N}-\mathbf{1})\}] \tag{5}$$

## 3 Computation of the KL Spectrum

This section describes the computation of the KL spectrum of a discrete MVL function. Specifically, the KL spectrum of a discrete function is computed using the definition as provided in the previous section followed by a discussion of the relationship with the spectrum of a Cayley graph.

In classical processing of signals arising from physical processes, the Wiener-Khinchine theorem gives the relationship between the autocorrelation and the power spectral density of a signal. The Wiener-Khinchine theorem states that the power spectral density of a physical signal is equivalent to the Fourier transform of the signal's autocorrelation function [17].

In the work described here, we are interested in a different aspect of the autocorrelation function. Instead of computing the spectrum of the autocorrelation function, we examine the spectrum of the autocorrelation matrix. In [1], a Cayley graph is formulated using the additive Abelian group and the generator function $e = f(m_i \oplus m_j)$ where $m_i$ is a minterm of the binary-valued function $f$ and $e$ is the edge in the Cayley graph adjacent to vertices of the group elements $m_i$ and $m_j$. In [1] it is also proven that the spectrum of this Cayley graph is equivalent to the Walsh transform of the binary-valued function $f$. This result is generalized to functions depending on discrete non-binary valued variables in [16] and to the class of mixed-radix functions in [15]. This leads to Observation 1 where the Abelian group consists of all possible minterms and the group operator is $\ominus$.

**Observation 1** *The autocorrelation matrix as defined in Equation 5 is equivalent to the adjacency matrices of the Cayley graphs described in [1, 15, 16].*

If the $\ominus$ operator in the autocorrelation definition given in Equation 3 is replaced with the $\circ$ operation, the corresponding autocorrelation matrix, $\mathbf{F}_\circ$ becomes that shown in Equation 6.

$$\mathbf{F}_\circ = [\mathbf{f}(\mathbf{x} \circ \mathbf{0}) \;\; \mathbf{f}(\mathbf{x} \circ \mathbf{1}) \; \ldots \; \mathbf{f}\{\mathbf{x} \circ (\mathbf{N}-\mathbf{1})\}] \tag{6}$$

The extension of Obervation 1 to the Cayley graph based on the symmetric group $S$ with the generator given in Equation 7 leads to the result stated in Observation 2.

$$e = f(s_i \circ s_j) \tag{7}$$

**Observation 2** *The autocorrelation matrix* $\mathbf{F}_\circ$ *given in Equation 6 is equivalent to the adjacency matrix of a Cayley graph defined over the symmetric group $S$ using the generator given in Equation 7.*

Because the KL spectrum is defined as the set of eigenvalues of the autocorrelation matrix, and by Observation 2, this matrix is equivalent to that describing a particular Cayley graph, it is seen that the Cayley graph spectrum is in fact the KL spectrum of the function $f$.

## 4 Example KL Spectrum Calculations

As an example of the calculation of the KL spectrum of a discrete MVL function, consider the mixed-radix, binary-valued function defined in truth-table form in Table 3. Note that this function can be considered to have a support variable set consisting of the binary-valued variable $x_1$ and the ternary-valued variable $x_2$, or, the single sextary-valued variable $X$. Also, a mapping is assigned to each domain value of $f$ in Table 3 although this is done arbitrarily. The general form of the adjacency matrix of the Cayley graph for functions with this set of domain points is given in Equation 8.

$$\mathbf{A} = \begin{bmatrix} f(0) & f(1) & f(2) & f(3) & f(4) & f(5) \\ f(1) & f(2) & f(0) & f(4) & f(5) & f(3) \\ f(2) & f(0) & f(1) & f(5) & f(3) & f(4) \\ f(3) & f(5) & f(4) & f(0) & f(2) & f(1) \\ f(4) & f(3) & f(5) & f(1) & f(0) & f(2) \\ f(5) & f(4) & f(3) & f(2) & f(1) & f(0) \end{bmatrix} \quad (8)$$

**Table 3. Truth-table of Example Function for Computation of the KL Spectrum**

| $x_1$ | $x_2$ | $X$ | *Mapping* | $f$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 |
| 0 | 2 | 2 | 2 | 1 |
| 1 | 0 | 3 | 3 | 0 |
| 1 | 1 | 4 | 4 | 1 |
| 1 | 2 | 5 | 5 | 0 |

The vertex set of $Cay(V, E)$ is $V = \{0, 1, 2, 3, 4, 5\}$. The edge set is given by the generator in Equation 7 which consists of the ordered pairs $E = \{(0, 1), (0, 2), (0, 4), (1, 0), (1, 1), (1, 3), (2, 0), (2, 2), (2, 5), (3, 2), (3, 4), (3, 5), (4, 0), (4, 3), (4, 5), (5, 1), (5, 3), (5, 4)\}$. The adjacency matrix describing this $Cay(V, E)$ is given by the matrix $\mathbf{A}$ in Equation 9 and the characteristic equation $C(\lambda)$ of $\mathbf{A}$ is given in Equation 10.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \quad (9)$$

$$C(\lambda) = \lambda^6 - 2\lambda^5 - 5\lambda^4 + 6\lambda^3 \quad (10)$$

The spectrum of $Cay(V, E)$ and corresponding KL spectrum of $f(X)$ becomes the set of roots of $C(\lambda)$. Finding the roots of Equation 10 results in the set of spectral values $\lambda_i \in \Lambda$.

$$\Lambda = \{0, 0, 0, 1, -2, 3\} \quad (11)$$

To illustrate the importance of mapping the domain values of the function of interest to the vertices of the Cayley graph, consider the alternative mapping of the same example function as shown in Table 4.

**Table 4. Truth-table of Example Function for Computation of the KL Spectrum with Alternative Mapping**

| $x_1$ | $x_2$ | $X$ | *Mapping* | $f$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 3 | 0 |
| 0 | 2 | 2 | 4 | 0 |
| 1 | 0 | 3 | 1 | 1 |
| 1 | 1 | 4 | 5 | 1 |
| 1 | 2 | 5 | 2 | 1 |

The corresponding adjacency matrix for this alternatively mapped function is given in Equation 12. The characteristic equation and eigenvalues (or function KL spectral coefficients) are given in Equations 13 and 14.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (12)$$

$$C(\lambda) = \lambda^6 - 9\lambda^4 \quad (13)$$

$$\Lambda = \{0, 0, 0, 0, -3, 3\} \quad (14)$$

These two examples illustrate the important fact that although the KL spectra are proven to give a maximum number of zero-valued coefficients, the arbitrary mapping of the elements of the $S_3$ group members to the minterms of the function $f$ also affect the number of zero-elements. In future work, it is of interest to determine the complexity of finding mappings that yield the "best" KL spectrum in terms of maximizing the number of zeros in the resulting spectrum and to perhaps develop techniques to find good mappings.

## 5 Conclusion

It has been shown that the spectrum of a Cayley graph over the symmetry group with a particular generator based on a function of interest is equivalent to the KL spectrum when the autocorrelation is computed using the group product operator $\circ$. In addition to the theoretical interest in relating the KL spectrum to algebraic graph techniques, this result can have useful applications with regard to MVL function estimation techniques and lossless compression of MVL functions.

In the future it is planned to investigate the application of these techniques for generating compact decision diagram representations of discrete MVL functions. By representing functions in the KL spectral domain, an extension of the use of *zero-suppressed binary decision diagrams* (ZBDDS) [13] to the multiple-valued case may lead to compact representations of this class of functions. It is also planned to investigate the use of these methods to estimate a function $f$ with a dependent support set of cardinality $n$ with a corresponding estimated function $\hat{f}$ that has a dependent variable set of cardinality $m$ where $m < n$.

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